Big Pure-projective modules

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Overview

General aim:
Study the direct summands of $M^{(I)}$ where M is a finitely generated module over a commutative local noetherian ring R and for arbitrary I. We can reduce to I countable.

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Can we construct modules in $\mathrm{Add}(M)$ that are not direct sum of finitely generated ones?YES!!!

The monoid language

S ring, associative with unit

$V(S) =$ set of isomorphism classes of finitely generated projective right S-modules.

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$$V(M) \cong V(S) \qquad \text{and} \qquad V^*(M) \cong V^*(S)$$

The dimension function for semilocal rings

A ring S is semilocal if modulo its Jacobson radical is semisimple artinian . That is $S/J(S) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ with $D_1, \dots D_k$ division rings.

Fix an ordered set $V_1, \dots, V_k$ of representatives of the isomorphism classes of the simple right S-modules such that $\mathrm{End}_S(V_i) \cong D_i$ for $i = 1, \dots, k$.

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We associate to the isomorphism class of P its dimension

$$\dim(\langle P \rangle) = (a_1, \ldots, a_k) \in (\mathbb{N}_0 \cup \{\infty\})^k = (\mathbb{N}_0^*)^k$$

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- if I_i is finite, a_i is the cardinality of I_i
- $a_i = \infty$ otherwise.

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Still S is a general semilocal ring.....

$$\dim\colon V(S) \to \mathbb{N}_0^k$$

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is a monoid morphism, which is injective!!! (Prihoda 2007)

What is $\dim(V^*(S))$? We do not know, in general

The case of non necessarily commutative noetherian rings

Theorem

(H., Prihoda 2010) Let A be a submonoid of $(\mathbb{N}_0^*)^k$ containing $(n_1, \ldots, n_k) \in \mathbb{N}^k$. Then the following statements are equivalent:

(1) *A is is the set of solutions in $\mathbb{N}_0^*$ of a system of homogeneous diophantic linear equations and of congruences*

$$E_1 \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = E_2 \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \qquad \text{and} \qquad D \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix}$$

where the coefficients of the matrices D, E_1 and E_2 as well as $m_1, \ldots, m_n$ are elements of $\mathbb{N}_0$ and $m_i > 1$.

(2) *There exist a noetherian semilocal ring S with $\dim(V^*(S)) = A$.*

In the above situation, $\dim(V(S)) = A \cap \mathbb{N}_0^k$.

Solving systems in $\mathbb{N}_0^*$ has some surprises...

Some examples with $k = 2$:

- The solutions of $\boxed{x = y}$ in $\mathbb{N}_0$ are $B = \{(n, n) \mid n \in \mathbb{N}_0\}$ and the solutions in $\mathbb{N}_0^*$ are $A_1 = B \bigcup \{(\infty, \infty)\} = B + \infty \cdot B$

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- Fix $m > 1$. The solutions of $\boxed{2x + y = 2y + x \text{ and } x + my \in m\mathbb{N}_0^*}$ are

$$A_4 = mA_1 + (\infty, 0)\mathbb{N}_0 + (0, \infty)\mathbb{N}_0 + (\infty, 1)\mathbb{N}_0 + (1, \infty)\mathbb{N}_0$$

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Finitely generated modules

Let R be a local commutative noetherian ring, with completion $\hat{R}$. Let M_R be a finitely generated right R-module with endomorphism ring S. Then

$$M \otimes_R \hat{R} \cong L_1^{n_1} \oplus \cdots \oplus L_k^{n_k}$$

with $L_1, \ldots, L_k$ indecomposable $\hat{R}$-modules.
If N is a countably generated module in $\mathrm{Add}\,(M)$ we set

$$\dim(\langle N \rangle) = \dim(\langle \mathrm{Hom}_R(M, N) \rangle) = (a_1, \ldots, a_k)$$

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$\dim(V(M))$ was determined by Roger Wiegand (2001) :

- Diophantic monoids for torsion free modules over analitically unramified local domains of Krull dimension 1.
- The general case for modules over local domains of Krull dimension 2

Main realization Theorem

Theorem

Consider submonoid of $\mathbb{N}_0^k$, containing $(n_1, \ldots, n_k) \in \mathbb{N}^k$, defined by a system of equations $E_1\mathbf{X} = E_2\mathbf{X}$ where E_i have entries in $\mathbb{N}_0$. Set F be a matrix of suitable size which all its entries 1.

Then there exists a local noetherian domain R of Krull dimension 1 with reduced completion $\hat{R}$, and a finitely generated torsion free R-module M such that $\dim_R(V^*(M))$ is the set of solutions of the system

$$(E_1 + F)\mathbf{X} = (E_2 + F)\mathbf{X}.$$

Here, any module of the form $L_1^{(a_1)} \oplus \cdots \oplus L_k^{(a_k)}$ is extended from an R-module provided that at least one a_i is infinite

Thanks for your kind attention!!!