Ends of moduli spaces of Higgs bundles

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Special Session *Higgs Bundles and Character Varieties*
Joint International Meeting of the AMS, EMS and SPM

Porto, June 2015

Hitchin's equation

Setting

X compact Riemann surface, $\pi : E \to X$ complex rank-2 vector bundle

- Auxiliary data: g compatible Riemannian metric on X, h hermitian metric on E
- Fixed determinant case: A_0 fixed unitary connection on E, consider unitary connections of the form

$$A = A_0 + \alpha, \quad \alpha \in \Omega^1(\mathfrak{su}(E))$$

  and trace-free Higgs-field $\Phi \in \Omega^{1,0}(\mathfrak{sl}(E))$

- Hitchin's equation

$$F_A^\perp + [\Phi \wedge \Phi^*] = 0, \quad \bar{\partial}_A \Phi = 0$$

  where $F_A^\perp$ is the trace-free part of the curvature

Hitchin's equation

Basic question

Consider sequence (A_n, Φ_n) of solutions

- $\|\Phi_n\|_{L^2} \leq C < \infty$: Uhlenbeck compactness $\implies (A_n, \Phi_n)$ subconverges to solution (A_∞, Φ_∞)
- $\|\Phi_n\|_{L^2} \to \infty$: (A_n, Φ_n) exiting end of the moduli space

Question

What is the degeneration behavior of a diverging sequence of solutions?

Ultimate goals:

- Describe asymptotics of Hyperkahler metric
- Compute space of L^2-harmonic forms

Limiting configurations

The limiting fiducial solution

Consider trivial rank-2 vector bundle over $\mathbb{C}$ and the Higgs field

$$\Phi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz, \quad \Longrightarrow \det \Phi = -z dz^2.$$

Goal: Find hermitian metric H_∞ on $\mathbb{C}^\times$ such that

$$\bar{\partial}(H_\infty^{-1} \partial H_\infty) = 0, \quad [\Phi \wedge \Phi^{*_{H_\infty}}] = 0.$$

Ansatz: Rotationally symmetric

$$H_\infty = \begin{pmatrix} \alpha(r) & b(r) \\ \bar{b}(r) & \beta(r) \end{pmatrix}$$

with α, β real valued, $\alpha > 0$ and $\alpha\beta - |b|^2 = 1$.

Limiting configurations

The (limiting) fiducial solution

Short Calculation $\Longrightarrow$

The unique solution is given by

$$H_\infty = \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix}$$

and the corresponding pair

$$A_\infty^{fid} := \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right), \quad \Phi_\infty^{fid} := \begin{pmatrix} 0 & r^{1/2} \\ zr^{-1/2} & 0 \end{pmatrix} dz$$

solves the decoupled equation

$$F_{A_\infty} = 0, \quad [\Phi_\infty \wedge (\Phi_\infty)^*] = 0, \quad \bar{\partial}_{A_\infty}\Phi_\infty = 0$$

on $\mathbb{C}^\times$.

Limiting configurations

Globally

Fix $q \in H^0(K_X^2)$ with simple zeroes. Let $X^\times = X \setminus q^{-1}(0)$.

A *limiting configuration* associated with q is a pair (A_∞, Φ_∞) on $X^\times$ such that

- (A_∞, Φ_∞) solves

$$F_{A_\infty}^\perp = 0, \quad [\Phi_\infty \wedge \Phi_\infty^*] = 0, \quad \bar\partial_{A_\infty} \Phi_\infty = 0$$

- $\det \Phi_\infty = q$
- $(A_\infty, \Phi_\infty) = (A_\infty^{fid}, \Phi_\infty^{fid})$ near each $p \in q^{-1}(0)$

after fixed choice of holomorphic coordinate and unitary frame.

Limiting configurations

Existence

Fix $q \in H^0(K_X^2)$ with simple zeroes. Let $X^\times = X \setminus q^{-1}(0)$.

Theorem (MSWW)

For each pair (A, Φ) with $\bar\partial_A \Phi = 0$ and $\det \Phi = q$ there exists a complex gauge transformation g_∞ on $X^\times$ such that $(A, \Phi)^{g_\infty}$ is a limiting configuration.

Note: $\det \Phi$ simple zeroes $\Longrightarrow (\bar\partial_A, \Phi)$ stable Higgs bundle

Proof:

- Normalize the Higgs field on $X^\times$
- Gauge away the curvature

Hitchin: Interpretation as parabolic Higgs bundles

Limiting configurations

Moduli space

Fix $q \in H^0(K_X^2)$ with simple zeroes.

$\mathcal{M}_\infty(q) :=$ space of limiting configurations associated with q

$\gamma :=$ genus of X

Theorem (MSWW)

$\mathcal{M}_\infty(q)$ is a torus of real dimension $6\gamma - 6$.

Note: generic fiber of Hitchin fibration Prym variety associated with q (= complex torus of dimension $3\gamma - 3$)

Hitchin: direct identification of $\mathcal{M}_\infty(q)$ with Prym variety

Desingularization

The (desingularized) fiducial solution

Now look for nonsingular solutions of Hitchin's equation

$$\bar{\partial}(H_t^{-1}\partial H_t) + t^2[\Phi \wedge \Phi^{*H_t}] = 0$$

on $\mathbb{C}$ for $t < \infty$. H_t rotationally symmetric $\Longrightarrow$

$$H_t = \begin{pmatrix} r^{1/2}e^{h_t(r)} & 0 \\ 0 & r^{-1/2}e^{-h_t(r)} \end{pmatrix}$$

where after substitution $h_t(r) = \psi(\rho)$ with $\rho = \frac{8}{3}tr^{3/2}$ and ψ solves Painlevé type III equation

$$\psi'' + \frac{\psi'}{\rho} = \frac{1}{2}\sinh(2\psi).$$

$\Longrightarrow \exists!$ solution h_t satisfying $h_t(r) + \frac{1}{2}\log(r) \to 0$ as $r \searrow 0$ and $h_t(r) \to 0$ as $r \nearrow \infty$.

Desingularization

The (desingularized) fiducial solution

The corresponding pair

$$A_t^{fid} = f_t(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right),$$

$$\Phi_t^{fid} = \begin{pmatrix} 0 & r^{1/2} e^{h_t(r)} \\ z r^{-1/2} e^{-h_t(r)} & 0 \end{pmatrix} dz$$

where $f_t(r) = \frac{1}{8} + \frac{1}{4} r \partial_r h_t$ solves Hitchin's equation

$$F_{A_t} + t^2 [\Phi_t \wedge \Phi_t^*] = 0, \quad \bar{\partial}_{A_t} \Phi_t = 0$$

on $\mathbb{C}$ (called the *fiducial solution* by Gaiotto, Moore and Neitzke).

Key properties:

- $(A_t^{fid}, \Phi_t^{fid})$ nonsingular on $\mathbb{C}$
- $(A_t^{fid}, \Phi_t^{fid}) \to (A_\infty^{fid}, \Phi_t^{fid})$ as $t \to \infty$ locally uniformly on $\mathbb{C}^\times$ and exponentially fast in t.

Desingularization

Globally

Theorem (MSWW)

For each limiting configuration (A_∞, Φ_∞) there exists a family (A_t, Φ_t) of solutions to Hitchin's equation

$$F_{A_t} + t^2[\Phi_t \wedge \Phi_t^*] = 0, \quad \bar{\partial}_{A_t}\Phi_t = 0$$

such that $(A_t, \Phi_t) \to (A_\infty, \Phi_\infty)$ as $t \to \infty$ locally uniformly on $X^\times$ and exponentially fast in t.

Proof:

- glue A_t^{fid} to A_∞ using partition of unity to obtain approximate solution
- deform into actual solution for large t

Work in progress

- multiple zeroes
- determination of asymptotics of Hyperkahler metric

  $$g_{HK} = g_{sf} + O(e^{-\delta t})$$

  where is g_{sf} is the so-called *semi-flat metric*
- higher rank (touches thesis work of Laura Fredrickson)