Tableaux in Brill-Noether theory

Nathan Pflueger (Brown University)

12 June 2015

Joint work with M. Chan (Harvard), A. López (IMPA), and M. Teixidor i Bigas (Tufts).

Brill-Noether varieties

$k = \overline{k}$ is an algebraically closed field.
g, r, d are positive integers.
C is a smooth projective curve of genus g.

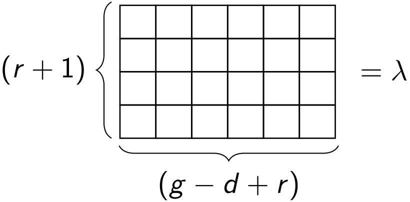
$$\begin{aligned} G^r_d(C) &= \{(L, V) : L \text{ a degree } d \text{ line bundle,} \\ &\quad V \subseteq H^0(\mathcal{L}) \text{ a } (r+1)\text{-dimensional vector space}\} \end{aligned}$$

The (closed) points are called *linear series.*

The Young diagram

$$G^r_d(C) = \{\text{linear series}(L, V)\}$$

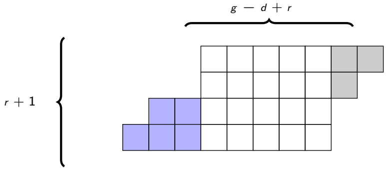
Associate to the integers g, r, d the following Young diagram.



Tableaux on λ will encode information about the geometry and topology of $G^r_d(C)$.

Arbitrary skew shapes

Almost everything I say generalizes to arbitrary *skew shapes*, such as the following.



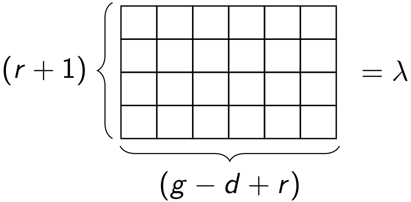
These describe varieties $G_d^{r,\alpha,\beta}(C, p, q)$ of linear series with specified ramification at p and q.
For simplicity, I will focus on rectangles in this talk.

General curves

Let C be a *general curve*.

Theorem (Griffiths-Harris, Gieseker)

1. *$G^r_d(C)$ is nonempty if and only if $|\lambda| \leq g$. If nonempty, it is a smooth variety of dimension $g - |\lambda|$.*
2. *If $|\lambda| = g$, then the number of elements of $G^r_d(C)$ is equal to the number of standard Young tableaux on λ.*



Example

A general genus 4 curve has two degree 3 maps to $\mathbb{P}^1$.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

(Here, $g = 4,\ d = 3,\ r = 1$, so $r + 1 = 2$ and $g - d + r = 2$.)

Example

A genus g curve has a unique canonical class.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & g \\ \hline \end{array} \quad \text{(transposed)}$$

(Here, $d = 2g - 2, r = g - 1$, so $r + 1 = g$ and $g - d + r = 1$.)

General curves

A suggestive reformulation of the enumerative Brill-Noether theorem: when $|\lambda| = g$,

$$\chi_{\text{top}}(G^r_d(C)) = \#\,(\text{standard tableaux on } \lambda)\,.$$

[1] Originally calculated by Eisenbud-Harris in the rectangular case, without reference to tableaux.

A suggestive reformulation of the enumerative Brill-Noether theorem: when $|\lambda| = g$,

$$\chi_{\text{top}}(G^r_d(C)) = \#\left(\text{standard tableaux on } \lambda\right).$$

Theorem (Chan, López, P., Teixidor[1])

When $|\lambda| = g - 1$ (so $G^r_d(C)$ is itself a smooth curve),

$$\chi_{\text{top}}(G^r_d(C)) = -\#(\text{“1-overstuffed” tableaux on } \lambda)$$

[1] Originally calculated by Eisenbud-Harris in the rectangular case, without reference to tableaux.

Overstuffed tableaux

A "1-overstuffed tableau" is a labeling of the boxes of λ with the numbers $\{1, 2, \cdots, |\lambda| + 1\}$ such that exactly one box has two labels (where order matters), and removing either one of these labels results in a tableaux with *strictly increasing* rows and columns.

Example

If $g = 5,\ d = 4,\ r = 1$, then the following are considered distinct 1-overstuffed tableaux.

1	24
3	5

1	42
3	5

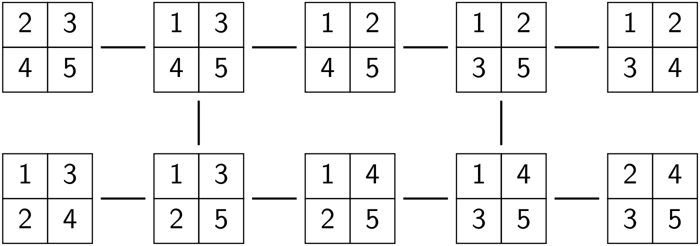
1	2
3	45

There are 17 others; the topological Euler characteristic of the curve $G^1_4(C)$ is -20.

The Brill-Noether graph

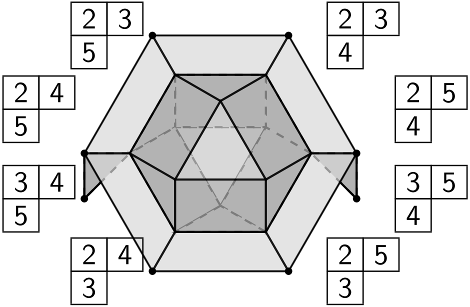
An equivalent formulation (used in our paper): when $G^r_d(C)$ is a curve,

$$\chi_{\text{top}}(G^r_d(C)) = 2 \cdot \#\,(\text{edges in the "Brill-Noether graph"})$$



Higher dimensions?

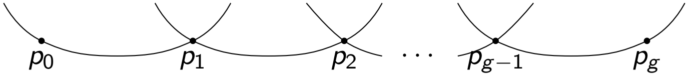
Overstuffed tableaux can similarly be used as the faces of higher-dimensional polyhedral complexes, such as the following.



In future work, we hope to read topological information about higher-dimensional Brill-Noether varieties from these polyhedral complexes.

The method

Let X be a chain of elliptic curves, as follows.

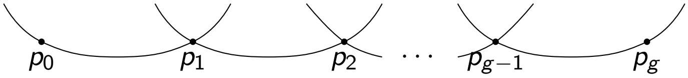


The theory of *limit linear series* (Eisenbud, Harris, Osserman) makes it possible to define a scheme

$$G^r_d(X).$$

This scheme is reducible; it's irreducible components (and their incidence relations) can be enumerated via certain tableaux on λ.

The method



Genericity assumption: Suppose that on the nth curve in the chain, the divisor $(p_n - p_{n-1})$ is non-torsion in the Jacobian.

Theorem (Chan,López,P.,Teixidor)

If $|\lambda| = g - 1$*, then*

1. $G^r_d(X)$ *is a reduced nodal curve. The stable model of this curve has dual graph equal to the Brill-Noether graph, and all irreducible components are of genus* 1*.*
2. $G^r_d(X)$ *is a flat limit of varieties* $G^r_d(C)$*, where* C *varies over a family of smooth curves degenerating to* X*.*

Background

- Elliptic chains specialize to "flag curves" (of Eisenbud-Harris). They have significant advantages in characteristic p.
- First use in Brill-Noether theory by Welters.
- Used to great effect by Teixidor in the context of higher-rank vector bundles.
- Our work builds off of a recent paper of Castorena-López-Teixidor dealing with the case $r = 1$.

What if there is torsion in the chain?

Example

Displacement tableaux give information about *special* curves.

1	2	3	5	6	10
2	4	6	8	9	13
3	5	7	10	12	14
7	8	9	11	13	15

$$\underline{m} = (0, 2, 4, 0, 4, 3, 3, 4, 4, 4, 0, 0, 3, 0, 0)$$
$$g = 15, \quad r = 3, \quad d = 12$$

What if there is torsion in the chain?

Example

Displacement tableaux give information about *special* curves.

1	2	3	5	6	10
2	4	6	8	9	13
3	5	7	10	12	14
7	8	9	11	13	15

$$\underline{m} = (0,2,4,0,4,3,3,4,4,4,0,0,3,0,0)$$
$$g = 15, \quad r = 3, \quad d = 12$$

$\Rightarrow$ There is an irreducible component of $\mathcal{G}^3_{12,15}$ (the relative version of $G^3_{12}(C)$), finite to one over a subvariety of $\mathcal{M}_{15}$ with codimension $|\lambda| - g = 9$.