A family of explicitly diagonalizable weighted Hankel matrices generalizing the Hilbert matrix

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Abstract

A three-parameter family $B = B(a, b, c)$ of weighted Hankel matrices (following Peller's terminology) is introduced,

$$B_{j,k} = \frac{\Gamma(j+k+a)}{\Gamma(j+k+b+c)} \sqrt{\frac{\Gamma(j+b)\Gamma(j+c)\Gamma(k+b)\Gamma(k+c)}{\Gamma(j+a)\, j!\, \Gamma(k+a)\, k!}}$$

$j, k \in \mathbb{Z}_+$,

$$a > 0,\ b > 0,\ c > 0,\ a < b + c,\ b < a + c,\ c \leq a + b$$

The structure of B: $B_{j,k} = w_j w_k h(j+k)$, where

$$w_j = \sqrt{\frac{\Gamma(j+b)\Gamma(j+c)}{\Gamma(j+a)\, j!}}, \quad h(z) = \frac{\Gamma(z+a)}{\Gamma(z+b+c)}$$

$\ast$ The Hilbert matrix is included as a particular case: $H(\theta) := B(\theta, \theta, 1)$,

$$H(\theta)_{j,k} = \frac{1}{j+k+\theta}, \quad j, k = 0, 1, 2, \ldots.$$

$*$ There exists a three-parameter family of real symmetric Jacobi matrices, $T(a,b,c)$, commuting with $B(a,b,c)$.
$*$ The orthogonal polynomials associated with $T(a,b,c)$ are the continuous dual Hahn polynomials.
$*$ Consequently, a unitary mapping U diagonalizing $B(a,b,c)$ can be constructed explicitly.
$*$ The spectrum is purely absolutely continuous filling the interval $[0, M(a,b,c)]$,

$$M(a,b,c) = \frac{1}{\Gamma(b+c-a)}\,\Gamma\left(\frac{b+c-a}{2}\right)^2$$

$*$ If the assumption $c \leq a+b$ is relaxed:

$$a>0,\ b>0,\ c>0,\ a<b+c,\ b<a+c,\ c>a+b,$$

the spectrum contains also a finite discrete part lying above $M(a,b,c)$.

A class of integral operators and the Mellin transform

Let K be an integral operator on $L^2((0,\infty),\mathrm{d}x)$,
its integral kernel $\mathcal{K}(x,y)$ be real, symmetric,
homogeneous of degree -1 and

$$\int_0^\infty |\mathcal{K}(t,1)|\, t^{-1/2}\mathrm{d}t < \infty$$

Then K is diagonalizable by the Mellin integral transform,
$K \sim$ the multiplication operator on $L^2(\mathbb{R},\mathrm{d}\xi)$ by the function

$$g(\xi) = \int_0^\infty \mathcal{K}(t,1)\, t^{-1/2-i\xi}\mathrm{d}t = \int_{\mathbb{R}} e^{-i\xi x}\mathcal{K}(e^{x/2},e^{-x/2})\,\mathrm{d}x$$

This is a consequence of the symmetry of K:

K commutes with the one-parameter unitary group of dilatation transformations on $\mathbb{R}_+$ generated by $D = x\mathrm{d}/\mathrm{d}x + 1/2$.

The kernel of the Mellin transform $=$ a family
of generalized eigenfunctions of D.

Example:

$$\mathcal{K}_\ell(x,y) = \frac{(xy)^{\ell/2}}{(x+y)^{\ell+1}}\,,\quad \ell > -1 \text{ a parameter}$$

The integral operator $K_\ell \sim$ the multiplication operator by

$$g(\xi) = \int_0^\infty t^{(\ell-1)/2+i\xi}(t+1)^{-\ell-1}\mathrm{d}t = \frac{1}{\Gamma(\ell+1)}\left|\Gamma\left(\frac{1}{2}(\ell+1)+i\xi\right)\right|^2$$

It is not straightforward to find an authentic discrete analog of K_ℓ .
Given a homogenous kernel of degree -1 one can always restrict the kernel to the lattice $(\theta+\mathbb{Z}_+)\times(\theta+\mathbb{Z}_+)$ for some $\theta > 0$
$\Longrightarrow$ a semi-infinite matrix.

Example: $\mathcal{K}_\ell$, with $\ell = 0$, yields the (generalized) Hilbert matrix.

But as emphasized by Kato in one of his papers:
it appears that there may be inconveniences in applying to matrices some methods originally invented for integral operators.
In particular, no obvious discrete analog of the Mellin transform is at our disposal.

A discrete dilatation operator commuting with B

Let us decompose

$$\ell^2(\mathbb{Z}_+) = \ell^2(2\mathbb{Z}_+) \oplus \ell^2(2\mathbb{Z}_+ + 1)$$

With respect to the decomposition, let us introduce

$$A = \begin{pmatrix} B(a,b,c) & 0 \\ 0 & B(a+1,b+1,c) \end{pmatrix}, \ D = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix},$$

where

$$C_{j,j} = d(2j),\ C_{j+1,j} = -d(2j+1),\ \ C_{j,k} = 0 \text{ otherwise}$$

$$\begin{aligned} d(2j) &= \sqrt{(j+a)(j+b)}, \\ d(2j+1) &= \sqrt{(j+1)(j+c)}, \quad j,k = 0,1,2,\ldots \end{aligned}$$

This means $A = B(a,b,c) \oplus B(a+1,b+1,c)$;
$D = D(a,b,c)$ as a matrix operator on $\ell^2(\mathbb{Z}_+)$ fulfills

$$D_{j,j+1} = -D_{j+1,j} = d(j),\ \ D_{j,k} = 0 \text{ otherwise};\ \ d(j) \sim j \ \text{ for } j \gg 1$$

D can be regarded as a discrete analog of the dilatation operator.

Lemma

A and D commute.

Corollary

Let $T = T(a,b,c)$ be the three-parameter family of symmetric Jacobi matrices,

$$\begin{aligned} T_{j,j} &= j\,(j+c-1) + (j+a)(j+b), \\ T_{j,j+1} &= T_{j+1,j} = -\sqrt{(j+1)(j+a)(j+b)(j+c)}\,, \\ T_{j,k} &= 0 \ \ \textit{otherwise} \end{aligned}$$

Then $B(a,b,c)$ and $T(a,b,c)$ commute.

Proof. A and D^2 commute $\implies$ B and $T = CC^T$ commute.

Sketch of the proof of the lemma

$$AD - DA = 0 \iff B(a,b,c)\,C = C\,B(a+1,b+1,c)$$

Write $B(a,b,c) = WHW$, W is a diagonal, H is a Hankel matrix,

$$W_{j,j} = \sqrt{\frac{\Gamma(j+b)\Gamma(j+c)}{\Gamma(j+a)\,j!}},\ H_{j,k} = h(j+k),\ h(z) = \frac{\Gamma(z+a)}{\Gamma(z+b+c)}$$

Similarly, $B(a+1,b+1,c) = \tilde{W}\tilde{H}\tilde{W}$. Then $AD - DA = 0 \iff$

$$HV = \tilde{V}\tilde{H},\ \text{with } V = WC\tilde{W}^{-1},\ \tilde{V} = W^{-1}C\tilde{W}.$$

One finds that

$$V_{j,j} = j+a,\ V_{j+1,j} = -j-c,\ \tilde{V}_{j,j} = j+b,\ \tilde{V}_{j+1,j} = -j-1,$$

$V_{j,k} = \tilde{V}_{j,k} = 0$ otherwise.
Consequently, $AD - DA = 0$ is equivalent to

$$h(j+k)\,(j+k+a) = \tilde{h}(j+k)\,(j+k+b+c),\ \ \forall j,k$$

This is so, indeed,

$$\tilde{h}(z) = h(z+1) = \frac{z+a}{z+b+c}\,h(z)$$

The orthogonal polynomials associated with $T(a,b,c)$

The monic orthogonal polynomials associated with a Jacobi matrix T are defined by the recurrence: $P_{-1}(x)=0$, $P_0(x)=1$,

$$P_{j+1}(x)=\left(x-T_{j+1,j+1}\right)P_j(x)-\left(T_{j,j+1}\right)^2P_{j-1}(x).$$

In our case, we add to T a multiple of the unit operator:

$$\begin{aligned}T_{j,j} &= j(j+c-1)+(j+a)(j+b)-\frac{1}{4}(a+b-c)^2\\ &= -j\,(j+1)+(j-b+d)(j-c+d)+(j-a+d)(j-b+d)\\ &\quad +(j-a+d)(j-c+d)\end{aligned}$$

where $d=(a+b+c)/2$.

The associated monic orthogonal polynomials, $P_n(x)$, coincide with the continuous dual Hahn polynomials

$$\begin{aligned}P_n(x^2) &= (-1)^n\,S_n\!\left(x^2;\frac{b+c-a}{2},\frac{a+c-b}{2},\frac{a+b-c}{2}\right)\\ &= (-1)^n(b)_n(c)_n\,{}_3F_2\!\left(-n,\frac{b+c-a}{2}+ix,\frac{b+c-a}{2}-ix;b,c;1\right)\end{aligned}$$

The measure of orthogonality for the continuous dual Hahn polynomials: for $\alpha > 0$, $\beta > 0$, $\gamma > 0$,

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\alpha+ix)\Gamma(\beta+ix)\Gamma(\gamma+ix)}{\Gamma(2ix)}\right|^2 S_m(x^2;\alpha,\beta,\gamma)S_n(x^2;\alpha,\beta,\gamma)\,\mathrm{d}x$$

$$= \Gamma(n+\alpha+\beta)\Gamma(n+\alpha+\gamma)\Gamma(n+\beta+\gamma)\,n!\,\delta_{m,n}$$

For $\alpha < 0$, $\alpha+\beta > 0$, $\alpha+\gamma > 0$,

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\alpha+ix)\Gamma(\beta+ix)\Gamma(\gamma+ix)}{\Gamma(2ix)}\right|^2 S_m(x^2;\alpha,\beta,\gamma)S_n(x^2;\alpha,\beta,\gamma)\,\mathrm{d}x$$

$$+\frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\gamma)\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)}{\Gamma(-2\alpha)}$$

$$\times\sum_{k\in\mathbb{Z}_+,\,\alpha+k<0}(-1)^k\frac{(2\alpha)_k(\alpha+1)_k(\alpha+\beta)_k(\alpha+\gamma)_k}{(\alpha)_k(\alpha-\beta+1)_k(\alpha-\gamma+1)_k\,k!}$$

$$\times S_m(-(\alpha+k)^2;\alpha,\beta,\gamma)S_n(-(\alpha+k)^2;\alpha,\beta,\gamma)$$

$$= \Gamma(n+\alpha+\beta)\Gamma(n+\alpha+\gamma)\Gamma(n+\beta+\gamma)\,n!\,\delta_{m,n}$$

Diagonalization of $B(a,b,c)$

The continuous dual Hahn polynomials form an OG basis in the L^2 space with the above described orthogonality measure. Suppose (for definiteness)

$$0 < b \leq c \quad \text{and} \quad 0 < a < b + c$$

Then $b < a + c$; $a + b - c$ can be both positive or negative. If $c > a + b$ put ($\lceil x \rceil =$ the ceiling of $x \in \mathbb{R}$)

$$N(a,b,c) = \lceil (c - a - b)/2 \rceil - 1$$

Let $\mathcal{M}(a,b,c) = (0,+\infty)$ if $c \leq a + b$,

$$\begin{aligned} \mathcal{M}(a,b,c) &= (0,+\infty) \cup \{\lambda_k;\ k = 0,1,\ldots,N(a,b,c)\}, \\ \lambda_k &= i\left(\frac{a+b-c}{2} + k\right), \end{aligned}$$

if $c > a + b$.

$\mathrm{d}\mu =$ a positive Borel measure on $\mathcal{M}(a,b,c)$,

$$d\mu(x) = \rho(x)dx \quad \text{on} \quad (0, +\infty),$$

$$\rho(x) = \frac{x \sinh(2\pi x)}{\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)} \times \left| \Gamma\left(\frac{b+c-a}{2} + ix\right) \Gamma\left(\frac{a+c-b}{2} + ix\right) \Gamma\left(\frac{a+b-c}{2} + ix\right) \right|^2$$

and

$$\mu(\{\lambda_k\}) = \frac{(-1)^k \Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)} \left(1 + \frac{2k}{a+b-c}\right) \times \frac{(a+b-c)_k (a)_k (b)_k}{(a-c+1)_k (b-c+1)_k \, k!}$$

Theorem

The matrix operator $B(a,b,c)$ on $\ell^2(\mathbb{Z}_+)$ is unitarily equivalent to the multiplication operator by the function

$$\mathbf{h}(x) = \frac{1}{\Gamma(b+c-a)} \Gamma\left(\frac{b+c-a}{2} + ix\right) \Gamma\left(\frac{b+c-a}{2} - ix\right)$$

acting on $L^2(\mathcal{M}(a,b,c), d\mu)$.

The spectrum of $B(a, b, c)$

Corollary

The absolutely continuous part of the spectrum of $B(a, b, c)$ is simple and fills the interval $[0, M(a, b, c)]$ where

$$M(a,b,c) = \frac{1}{\Gamma(b+c-a)}\,\Gamma\left(\frac{b+c-a}{2}\right)^2$$

Corollary

If $c \le a + b$, the point spectrum of $B(a, b, c)$ is empty.
If $c > a + b$,

$$\mathrm{spec}_p\, B(a,b,c) = \{\beta_0, \beta_1, \ldots, \beta_{N(a,b,c)}\},$$

$$\beta_k := h(\lambda_k) = \frac{\Gamma(b+k)\Gamma(c-a-k)}{\Gamma(b+c-a)}$$

and one has $\beta_0 > \beta_1 > \ldots > \beta_{N(a,b,c)} > M(a,b,c)$.
In particular, all eigenvalues are simple.

Remark

For an eigenvector corresponding to β_k one can choose the vector v_k,

$$\langle e_n, v_k \rangle = \hat{P}_n(\lambda_k^2) = \sqrt{\frac{(b)_n (c)_n}{(a)_n \, n!}} \; {}_3F_2(-n, b+k, c-a-k; b, c; 1)$$

One has

$$\|v_k\|^2 = \frac{\Gamma(c)\Gamma(c-a-b-k+1)\, k!}{(c-a-b-2k)\Gamma(c-a-k)\Gamma(c-b-k)(a)_k(b)_k}$$

Hilbert's matrix

$H(\theta) := B(\theta, \theta, 1)$ is the generalized Hilbert matrix,

$$H(\theta)_{j,k} = \frac{1}{j+k+\theta}, \quad j,k = 0,1,2,\ldots.$$

By our assumptions, θ is positive.
By Corollaries above:

$*$ the absolutely continuous spectrum of $H(\theta)$ is simple filling the interval $[0,\pi]$ (irrespectively of θ),

$*$ the point spectrum is nonempty if and only if $0 < \theta < 1/2$,
if so it consists of the single simple eigenvalue $\beta_0 = \pi/\sin(\pi\theta)$.

The defining expression for $H(\theta)$ is free of square roots,
the range of θ can be extended to $\theta \in \mathbb{R}\backslash(-\mathbb{Z}_+)$.
The diagonalization method, as exposed above, can be applied to $H(\theta)$, with the extended range, without essential modifications.

$H(\theta)$ commutes with the Jacobi matrix $T(\theta)$,

$$\begin{aligned} T(\theta)_{j,j} &= 2j\,(j+\theta) - 1/4 + \theta, \\ T(\theta)_{j,j+1} &= T(\theta)_{j+1,j} = -(j+1)(j+\theta), \\ T(\theta)_{j,k} &= 0 \ \text{ otherwise}, \quad j,k = 0,1,2,\ldots \end{aligned}$$

The associated normalized orthogonal polynomials are given by

$$\begin{aligned} \hat{P}_n(x^2) &= \frac{1}{n!\,(\theta)_n}\, S_n\!\left(x^2; -\frac{1}{2}+\theta, \frac{1}{2}, \frac{1}{2}\right) \\ &= \frac{(\theta)_n}{n!}\; {}_3F_2\!\left(-n, -\frac{1}{2}+\theta+ix, -\frac{1}{2}+\theta-ix; \theta, \theta; 1\right). \end{aligned}$$

$\hat{P}_n(x^2)$ can also be expressed in terms of the Wilson polynomials,

$$\hat{P}_n(x^2) = \frac{4^n}{n!\,(\theta)_{2n}}\, W_n\!\left(\frac{x^2}{4}; -\frac{1}{4}+\frac{\theta}{2}, \frac{1}{4}, \frac{1}{4}+\frac{\theta}{2}, \frac{3}{4}\right),$$

$$\begin{aligned} &\frac{W_n(x^2;\alpha,\beta,\gamma,\delta)}{(\alpha+\beta)_n(\alpha+\gamma)_n(\alpha+\delta)_n} \\ &= {}_4F_3(-n, n+\alpha+\beta+\gamma+\delta-1, \alpha+ix, \alpha-ix; \alpha+\beta, \alpha+\gamma, \alpha+\delta; 1) \end{aligned}$$

If $\theta < 1/2$ then the orthogonality relation reads

$$\int_0^\infty \hat{P}_m(x^2)\hat{P}_n(x^2)\rho(x)\,\mathrm{d}x + \sum_{k=0}^{N(\theta)} \mu(\{\lambda_k\})\,\hat{P}_m(\lambda_k^2)\hat{P}_n(\lambda_k^2) = \delta_{m,n}$$

where

$$N(\theta) = \lceil -1/2 - \theta \rceil$$

$$\rho(x) = \frac{2x\tanh(\pi x)}{\Gamma(\theta)^2}\left|\Gamma\left(-\frac{1}{2} + \theta + ix\right)\right|^2$$

and

$$\lambda_k = i\left(-\frac{1}{2} + \theta + k\right),\quad \mu(\{\lambda_k\}) = \frac{\Gamma(1-\theta)^2\,(1-2\theta-2k)}{k!\,\Gamma(2-2\theta-k)}$$

The sum on the LHS is absent if $\theta \geq 1/2$.
$\{\hat{P}_n(x^2)\}$ is an ON basis in the corresponding L^2 space.

Theorem (Rosenblum, 1958)

For all real θ, $\theta \neq 0, -1, -2, \ldots$, *the singular continuous spectrum of* $H(\theta)$ *is empty, the absolutely continuous spectrum is simple filling the interval* $[0, \pi]$.
For $\theta \geq 1/2$, *the point spectrum of* $H(\theta)$ *is empty.*
For $\theta < 1/2$, *let* $N(\theta) = \lceil -1/2 - \theta \rceil$.
The only eigenvalues are $\pi / \sin(\pi\theta)$ *and* $-\pi / \sin(\pi\theta)$, *the multiplicities respectively equal* $N(\theta)/2 + 1$ *and* $N(\theta)/2$ *for* $N(\theta)$ *even, they are both equal to* $(N(\theta) + 1)/2$ *for* $N(\theta)$ *odd.*

The history of Hilbert's matrix:

- Hilbert's double series inequality, unpublished lectures: $H(1)$ is positive and bounded
- Hilbert's proof published by Weyl, Göttingen dissertation, 1908
- I. Schur: J. reine angew. Math. **140** (1911) 1-28: $\|H(1)\| = \pi$
- W. Magnus: Amer. J. Math. **72** (1950) 699-704: $\operatorname{spec} H(1) = [0, \pi]$ is purely continuous
- Marvin Rosenblum: Proc. Amer. Math. Soc. **9** (1958) 581-585: an explicit diagonalization of $H(\theta)$ for all $\theta \in \mathbb{R} \backslash (-\mathbb{Z}_+)$

THANK YOU FOR YOUR ATTENTION!