BV evolutions in phase field fracture

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- sharp crack (an example)
- phase field

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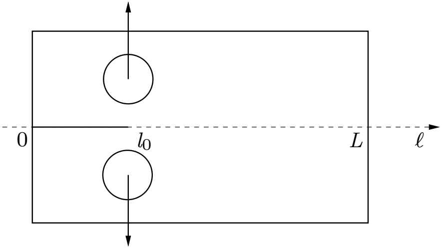
Setting and energy

EX: ASTM-Compact Tension



Admissible configurations $\mathcal{U}(t,\ell) = \{ u \in H^1(\Omega \setminus K_\ell) : u = \pm t \hat{e}_2 \text{ on } \partial_D \Omega \}$

Energy functional $\mathcal{F}(t,\ell) = \mathcal{E}(t,\ell) + G_c(\ell - l_0)$

Linear elastic energy $\mathcal{E}(t,\ell) = \min \left\{ \frac{1}{2} \int_{\Omega \setminus K_\ell} W(\boldsymbol{\epsilon}) \, dx : u \in \mathcal{U}(t,\ell) \right\}$

$\mathcal{E}(t,\cdot)$ is non-convex (in general)

Dissipated energy $\mathcal{D}(\ell) = G_c(\ell - l_0)$

Derivatives

The energy release $\boxed{G(t,\ell) = -\partial_\ell \mathcal{E}(t,\ell)}$ is Lipschitz continuous w.r.t. ℓ

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By irreversibility: configurations can be

$$\partial_\ell \mathcal{F}(t,\ell) \geq 0 \quad \Leftrightarrow \quad G(t,\ell) \leq G_c \quad \Leftrightarrow \quad |\partial_\ell \mathcal{F}(t,\ell)|^- = 0 \qquad \boxed{\text{equilibrium}}$$

$$\partial_\ell \mathcal{F}(t,\ell) < 0 \quad \Leftrightarrow \quad G(t,\ell) < G_c \quad \Leftrightarrow \quad |\partial_\ell \mathcal{F}(t,\ell)|^- > 0 \qquad \boxed{\text{instability}}$$

Time discrete propagation: incremental problem

Time discretization $t_k = k \triangle t$. Set $\ell(t_0) = l_0$.

Discrete evolution as a "gradient flow" [N.-Ortner (08)]

$$\begin{cases} l(0) = \ell(t_{k-1}) \\ l'(s) = |\partial_\ell \mathcal{F}\big(t_k, l(s)\big)|^- = |G(t_k, l(s)) - G_c|_+ \end{cases}$$

Then $\ell(t_k) = \lim_{s \to +\infty} l(s)$ is an equilibrium point for $\mathcal{F}(t_k, \cdot)$

"*the equilibrium position [...] must be one in which rupture of the solid has occurred if the system can pass from the unbroken to the broken condition by a process involving a continuous decrease of potential energy.*" [Griffith (20)]

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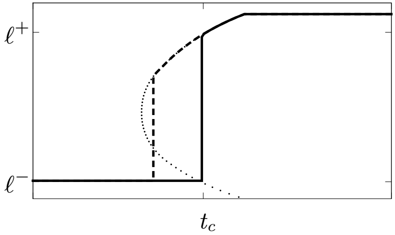
Define $\ell_{\triangle t}$ to be the piecewise affine interpolant of $\ell(t_k)$

Does $\ell_{\triangle t} \to \ell$ for $\triangle t \to 0$?

Which characterization for ℓ ?

A numerical example



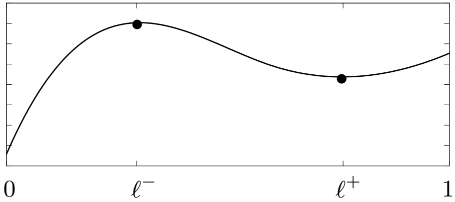
dotted: critical points $G(t,\ell) = G_c$
stable points $G(t,\ell) < G_c$ (left)
unstable points $G(t,\ell) > G_c$ (right)

bold: evolution $\ell(t)$ with $\ell_0 = 0.25$

dashed: energetic evolution

Energy $\mathcal{F}(t_c, \cdot)$ for $t_c \approx 0.95$

- $\mathcal{F}(t_c, \cdot)$ is non-convex
- $\mathcal{F}(t_c, \ell^+) < \mathcal{F}(t_c, \ell^-)$



Characterization of the evolution (I)

In terms of energy release G toughness G_c and by means of KT conditions.

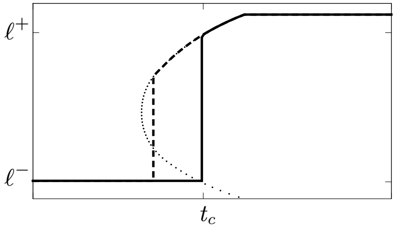
A non-decreasing $\ell \in BV(0,T)$ s.t. [N. Ortner (08)]

- $G(t,\ell^-(t)) \le G_c$ — equilibrium
- $\big(G(t,\ell^-(t)) - G_c\big)\, d\ell(t) = 0$ (in the sense of measures) — stability
- $G(t,s) \ge G_c$ for $s \in [\ell^-(t), \ell^+(t)]$ and $\boxed{t \in J(\ell)}$ — instability

"Same" as vanishing viscosity limit [Knees-Mielke-Zanini (08), N. (10)]

Jumps are characterized by unstable (catastrophic) propagation.

Graph parametrized BV-evolution



$t \mapsto \ell(t)$ in $BV(0,T)$

$s \mapsto (t(s), \ell(s))$ in $W^{1,\infty}(0,S)$

Parametrization of the extended graph

- continuity points if $t'(s) > 0$
- discontinuity intervals where $t'(s) = 0$

Characterization of the evolution (II)

A parametrization $s \mapsto (t(s), \ell(s))$ with $0 \le t' \le 1$ and $0 \le \ell' \le 1$ s.t.

- $|\partial_\ell \mathcal{F}(t(s), \ell(s))|^- = 0$ for every s with $t'(s) > 0$ — equilibrium

- $$\mathcal{F}(t(s), \ell(s)) = \mathcal{F}(0, \ell(0)) + \int_0^s \partial_t \mathcal{F}(t(\tau), \ell(\tau))\, t'(\tau)\, d\tau + - \int_0^s |\partial_\ell \mathcal{F}(t(\tau), \ell(\tau))|^- \, \ell'(\tau)\, d\tau$$ energy balance

cf. [Efendiev-Mielke (06), N. (14)]

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energy balance = stability + instability by the chain rule in BV

$$d\mathcal{F}(t, \ell(t)) = \partial_t \mathcal{F}(t, \ell(t))\, dt + \partial_\ell \mathcal{F}(t, \ell(t))\, d_{ac}\ell(t) + \sum_{t \in J(\ell)} [\![\mathcal{F}(t, \cdot)]\!]\, \delta_t$$

integration in time ... and comparison ...

Phase-field setting

Energy functional $\mathcal{F}(t,u,v) = \mathcal{E}(t,u,v) + G_c \mathcal{L}(v)$

$$\mathcal{U} = \{u \in H^1_0(\Omega)\} \qquad \mathcal{V} = \{v \in H^1 \, , 0 \le v \le 1\}$$

Linear elastic energy $\mathcal{E}(t,u,v) = \frac{1}{2} \int_\Omega (v^2 + \eta) \, W(\boldsymbol{\epsilon}) \, dx - \int_\Omega f(t) \cdot u \, dx$

Dissipated energy $\mathcal{L}(v) = \frac{1}{2} \int_\Omega (v-1)^2 + |\nabla v|^2 \, dx$

[Ambrosio-Tortorelli (90)]

$\mathcal{F}(t,\cdot,\cdot)$ is separately convex (quadratic)

Which quasi-static evolution ?

Time discrete evolution by alternate minimization

Time discretization $t_k = k\triangle t$. Set the i.c. $u(t_0)$ and $v(t_0)$

Define by recursion, $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\left\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\,\right\} \\ v_m \in \text{argmin}\left\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\right\} \end{cases}$$

Let $u(t_k) = \lim_{m\to+\infty} u_m$ and $v(t_k) = \lim_{m\to+\infty} v_m$

[Bourdin-Francfort-Marigo (00)]

Then $u(t_k), v(t_k)$ are equilibrium points for $\mathcal{F}(t_k, \cdot, \cdot)$ and separate minimizers

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Piecewise-affine interpolation $u_{\triangle t}$ and $v_{\triangle t}$

$u_{\triangle t}$ and $v_{\triangle t}$ do they converge and if so to which evolution ?

Minimization as a gradient flow: an example

Let $F(x) = \frac{1}{2} xAx + bx + c$ in $\mathbb{R}^n$ for $A^T = A > 0$ and $\|x\|_A = \sqrt{xAx}$

Step from x_0 to x_{min} minimizer of $F(x_0 + x') = F(x_0) + \nabla F(x_0)x' + \frac{1}{2} x'Ax'$

$$\nabla F(x_0) + Ax' = 0 \;\Leftrightarrow\; \boxed{\; x' = -\nabla_A F(x_0)}$$

where $dF(x_0)[\xi] = \langle \nabla_A F(x_0), \xi \rangle_A$

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By homogeneity $x(s) = x_0 - s\,\widehat{\nabla}_A F(x_0)$ solves

$$\boxed{x'(s) = -\widehat{\nabla}_A F(x(s)) = -\widehat{\nabla}_A F(x_0) \quad \text{for } 0 \le s \le \|\nabla_A F(x(s))\|_A}$$

• fixed descent direction • finite transition interval cf. $x'(\tau) = -\nabla F(x(\tau))$

Families of intrinsic norms for the phase field energy (I)

Write the energy

$$\mathcal{F}(t,u,v) = \tfrac{1}{2}\int_\Omega (v^2+\eta)\, W(\boldsymbol{\epsilon})\, dx - \int_\Omega f(t)\cdot u\, dx + G_c \mathcal{L}(v)$$
$$= \tfrac{1}{2}\|u\|_v^2 + \beta_t(u) + c_v$$

Hence $\partial_u \mathcal{F}(t,u,v)[\phi] = \langle u, \phi\rangle_v + \beta_t(\phi)$ for $v \in L^\infty(\Omega)$

The corresponding 'slope'

$$\left|\partial_u \mathcal{F}(t,u,v)\right|_v^- = \left|\min\left\{\partial_u \mathcal{F}(t,u,v)[\phi] \,:\, \|\phi\|_v \le 1\right\}\right|^-$$

If $t_n \to t$, $u_n \to u$ in $\mathcal{U}$ and $v_n \rightharpoonup v$ in $\mathcal{V}$ (with $0 \le v_n \le 1$) then

$$\liminf_n \left|\partial_u \mathcal{F}(t_n,u_n,v_n)\right|_{v_n}^- \ge \left|\partial_u \mathcal{F}(t,u,v)\right|_v^- .$$

Write the energy

$$\mathcal{F}(t,u,v) = \tfrac{1}{2}\int_\Omega v^2\, W(\boldsymbol{\epsilon})\, dx + \tfrac{1}{2} G_c \int_\Omega (v-1)^2 + |\nabla v|^2\, dx + c_{t,u}$$
$$= \tfrac{1}{2}\|v\|_u^2 + b(v) + c_{t,u}$$

Hence $\partial_v \mathcal{F}(t,u,v)[\xi] = \langle v, \xi\rangle_u + b(\xi)$ for every $\xi \in H^1(\Omega)$

If $u \in \operatorname{argmin}\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}$ then $u \in W^{1,p}(\Omega)$ with $p \gtrsim 2$ and $\Omega \subset \mathbb{R}^2$

[Herzog-Meyer-Wachsmuth (11) cf. also Knees-Rossi-Zanini (14)]

The corresponding 'unilateral slope'

$$\big|\partial_v \mathcal{F}(t,u,v)\big|_u^- = \Big|\min\big\{\partial_v \mathcal{F}(t,u,v)[\xi] \,:\, \xi \le 0\,, \|\xi\|_u \le 1\big\}\Big|^-$$

If $t_n \to t$, $u_n \to u$ in $W^{1,p}$ and $v_n \rightharpoonup v$ in $\mathcal{V}$, then

$$\liminf_n \big|\partial_v \mathcal{F}(t_n,u_n,v_n)\big|_{u_n}^- \ge \big|\partial_v \mathcal{F}(t,u,v)\big|_u^-$$

Back to alternate minimization

Define by recursion, $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\left\{ \mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U} \right\} \\ v_m \in \text{argmin}\left\{ \mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1} \right\} \end{cases}$$

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- In the limit configuration

discrete equilibrium

$$\big|\partial_u \mathcal{F}(t_k, u(t_k), v(t_k))\big|^-_{v(t_k)} = \big|\partial_v \mathcal{F}(t_k, u(t_k), v(t_k))\big|^-_{u(t_k)} = 0$$

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- For every m (minimization as gradient flow)

discrete energy balance

$$\begin{aligned} \mathcal{F}(t_k, u_{m+1}, v_{m+1}) = \mathcal{F}(t_k, u_m, v_m) \, + \\ & - \int_0^1 \big|\partial_u \mathcal{F}(t_k, u_{m+r}, v_m)\big|^-_{v_m} \, \| u_{m+1} - u_m \|_{v_m} \, dr \, + \\ & - \int_0^1 \big|\partial_z \mathcal{F}(t_k, u_{m+1}, v_{m+r})\big|^-_{u_{m+1}} \, \| z_{m+1} - z_m \|_{u_{m+1}} \, dr \end{aligned}$$

Arc-lenght parametrization

Finite length of the path (m and k) in $\mathcal{U}$ and $\mathcal{V}$

$$s_{k+1} = s_k + \sum_m \left(\| u_{m+1} - u_m \|_{H^1} + \| v_{m+1} - v_m \|_{H^1} \right) < S$$

Arc-length interpolation $t_{\triangle t}(s)$ and $u_{\triangle t}(s)$ etc.

$$[0,S] \ni s \mapsto (t_{\triangle t}(s), u_{\triangle t}(s), v_{\triangle t}(s)) \in [0,T] \times \mathcal{U} \times \mathcal{V}$$

Then $(t_{\triangle t}, u_{\triangle t}, v_{\triangle t})$ is bounded in $W^{1,\infty}([0,S]; [0,T] \times \mathcal{U} \times \mathcal{V})$

While $u_{\triangle t}$ belongs to $C([0,S]; W^{1,p})$

The limit evolution for $\triangle t \to 0$

A Lipschitz parametrization $s \mapsto (t(s), u(s), v(s))$ with [Knees-N. (..)]

$$0 \leq t'(s) \leq 1 \qquad v'(s) \leq 0$$

- for every s with $t'(s) > 0$ **equilibrium**

$$\left|\partial_u \mathcal{F}(t(s), u(s), v(s))\right|^{-}_{v(s)} = \left|\partial_v \mathcal{F}(t(s), u(s), v(s))\right|^{-}_{u(s)} = 0$$

- for every s **energy balance**

$$\begin{aligned} \mathcal{F}(t(s), u(s), v(s)) = \mathcal{F}(t(0), u(0), v(0)) + \int_0^s \partial_t \mathcal{F}(t(r), u(r), v(r))\, t'(r)\, dr \\ - \int_0^s \left|\partial_u \mathcal{F}(t(r), u(r), v(r))\right|^{-}_{v(r)} \|u'(r)\|_{v(r)}\, dr \\ - \int_0^s \left|\partial_v \mathcal{F}(t(r), u(r), v(r))\right|^{-}_{u(r)} \|v'(r)\|_{u(r)}\, dr \end{aligned}$$

Proof is based on [N. (14)] with "gradient flow arguments" e.g. [Sandier-Serfaty (04)]

Sketch of the proof (I)

$$\mathcal{F}_{\Delta t}(s) + \int_0^s \left|\partial_u \mathcal{F}_{\Delta t}(r)\right|^-_{v_{\Delta t}(r)} \| u'_{\Delta t}(r) \|_{v_{\Delta t}(r)} + \left|\partial_v \mathcal{F}_{\Delta t}(r)\right|^-_{u_{\Delta t}(r)} \| v'_{\Delta t}(r) \|_{u_{\Delta t}(r)} \, dr$$

$$\leq$$

$$\mathcal{F}_{\Delta t}(0) + \int_0^s \partial_t \mathcal{F}_{\Delta t}(r) \, t'_{\Delta t}(r) \, dr$$

Sketch of the proof (I)

$$\liminf_{\triangle t}$$

$$\mathcal{F}_{\triangle t}(s) + \int_0^s \left|\partial_u \mathcal{F}_{\triangle t}(r)\right|^-_{v_{\triangle t}(r)} \|u'_{\triangle t}(r)\|_{v_{\triangle t}(r)} + \left|\partial_v \mathcal{F}_{\triangle t}(r)\right|^-_{u_{\triangle t}(r)} \|v'_{\triangle t}(r)\|_{u_{\triangle t}(r)} \, dr$$

$$\leq$$

$$\limsup_{\triangle t} \; \mathcal{F}_{\triangle t}(0) + \int_0^s \partial_t \mathcal{F}_{\triangle t}(r) t'_{\triangle t}(r) \, dr$$

Sketch of the proof (I)

$$\mathcal{F}(s) + \int_0^s \big|\partial_u \mathcal{F}(r)\big|^-_{v(r)} \|u'(r)\|_{v(r)} + \big|\partial_v \mathcal{F}(r)\big|^-_{u(r)} \|v'(r)\|_{u(r)} \, dr$$

$$\text{[lsc]} \ \leq \ \liminf_{\triangle t}$$

$$\mathcal{F}_{\triangle t}(s) + \int_0^s \big|\partial_u \mathcal{F}_{\triangle t}(r)\big|^-_{v_{\triangle t}(r)} \|u'_{\triangle t}(r)\|_{v_{\triangle t}(r)} + \big|\partial_v \mathcal{F}_{\triangle t}(r)\big|^-_{u_{\triangle t}(r)} \|v'_{\triangle t}(r)\|_{u_{\triangle t}(r)} \, dr$$

$$\leq$$

$$\limsup_{\triangle t} \ \mathcal{F}_{\triangle t}(0) + \int_0^s \partial_t \mathcal{F}_{\triangle t}(r) \, t'_{\triangle t}(r) \, dr$$

Sketch of the proof (I)

$$\mathcal{F}(s) + \int_0^s \left|\partial_u \mathcal{F}(r)\right|^-_{v(r)} \|u'(r)\|_{v(r)} + \left|\partial_v \mathcal{F}(r)\right|^-_{u(r)} \|v'(r)\|_{u(r)} \, dr$$

[lsc] $\leq \liminf_{\Delta t}$

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$$\leq$$

$$\limsup_{\Delta t} \; \mathcal{F}_{\Delta t}(0) + \int_0^s \partial_t \mathcal{F}_{\Delta t}(r) t'_{\Delta t}(r) \, dr$$

$\leq$ [usc]

$$\mathcal{F}(0) + \int_0^s \partial_t \mathcal{F}(r) t'(r) \, dr$$

Sketch of the proof (II)

$$\mathcal{F}(s) - \mathcal{F}(0) \leq - \int_0^s \left|\partial_u \mathcal{F}(r)\right|^-_{v(r)} \|u'(r)\|_{v(r)} + \left|\partial_v \mathcal{F}(r)\right|^-_{u(r)} \|v'(r)\|_{u(r)} \, dr$$
$$+ \int_0^s \partial_t \mathcal{F}(r) t'(r) \, dr$$

Sketch of the proof (II)

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$$+ \int_0^s \partial_t \mathcal{F}(r) t'(r) \, dr$$

[chain rule] $\le \int_0^s \mathcal{F}'(r) \, dr$

Sketch of the proof (II)

$$
\begin{aligned}
\mathcal{F}(s) - \mathcal{F}(0) &\le -\int_0^s \big|\partial_u \mathcal{F}(r)\big|^-_{v(r)} \|u'(r)\|_{v(r)} + \big|\partial_v \mathcal{F}(r)\big|^-_{u(r)} \|v'(r)\|_{u(r)} \, dr \\
&\quad + \int_0^s \partial_t \mathcal{F}(r) t'(r) \, dr \\
\text{[chain rule]} \quad &\le \int_0^s \mathcal{F}'(r) \, dr \\
\text{[AC]} \quad &\le \mathcal{F}(s) - \mathcal{F}(0)
\end{aligned}
$$

All inequalities become equalities.

Continuity points: minimality?

Continuity points are "characterized" by $t'(s) > 0$

For $t'(s) > 0$

$$u(s) \in \mathrm{argmin}\big\{\mathcal{E}(t(s), \bullet, v(s)) : u \in \mathcal{U}\big\}$$

$$v(s) \in \mathrm{argmin}\big\{\mathcal{F}(t(s), u(s), \bullet) : v \in \mathcal{V}\,, v \le v(s)\big\}$$

In general separate minimality $\not\Rightarrow$ joint (global or local) minimality.

Continuity points: consistency with the constraint?

Possible thermodynamical inconsistency of the constraint: in general

$$v(s) \text{ decreasing } \not\Rightarrow \; G_c \mathcal{L}(v(s)) \text{ increasing}$$

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However if $t'(s) > 0$ then $\mathcal{L}'(v(s)) = \partial_v \mathcal{L}(v(s))[v'(s)] \geq 0$

thermodynamical consistency

By equilibrium and irreversibility

$$\partial_v \mathcal{E}\big(t(s), u(s), v(s)\big)[v'(s)] + G_c \, \partial_v \mathcal{L}(v(s))[v'(s)] \ \geq \ 0$$

$$\mathcal{E}(t, v, u) = \tfrac{1}{2} \int_\Omega v^2 \, W(\boldsymbol{\epsilon}) \, dx + c_{t,u}$$

[decreasing]

Continuity points: consistency with Griffith criterion?

Phase field energy release?

Given t, v let

$$\tilde{\mathcal{E}}(t,v) = \mathcal{E}(t, u_{t,v}, v) \quad \text{for} \quad u_{t,v} \in \text{argmin}\,\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}.$$

$$\partial_v \tilde{\mathcal{E}}(t,v)[\xi] = \int_\Omega v\,\xi\, W(\boldsymbol{\epsilon}_{t,v})\,dx = \partial_v \mathcal{E}(t, u_{t,v}, v)[\xi]$$

Normalized admissible variations $\quad \widehat{\Xi} = \{\xi \in H^1 : \xi \le 0 \text{ and } d\mathcal{L}(z)[\xi] = 1\}$

$$\mathcal{G}(t,v) = -\inf\left\{\partial_v \tilde{\mathcal{E}}(t,v)[\xi] : \xi \in \widehat{\Xi}\right\}$$

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Griffith's criterion in KT form. If $t'(s) > 0$ then

- $\mathcal{G}(t(s), v(s)) \le G_c$ — equilibrium
- $(\mathcal{G}(t(s), v(s)) - G_c)\,\mathcal{L}'(z(s)) = 0$ — stability

Discontinuity points: which gradient flow?

For simplicity consider $\|u'(s)\|_{v(s)} = \|v'(s)\|_{u(s)} = 1$ and

$$\big|\partial_u \mathcal{F}(t(s), u(s), v(s))\big|^-_{v(s)} = \big|\partial_v \mathcal{F}(t(s), u(s), v(s))\big|^-_{u(s)} = 1$$

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Steepest descent (normalized gradient flow)

$$u'(s) \in \text{argmin}\,\{\partial_u \mathcal{F}(t(s), u(s), v(s))[\phi] : \|\phi\|_{v(s)} = 1\}$$

$$\Updownarrow$$

$$\text{div}\left(\boldsymbol{\sigma}_{\text{pf}}(u(s) + u'(s))\right) = 0 \quad \text{in } H^{-1}(\Omega)$$

phase field visco-elastic flow

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