Genera of non-algebraic leaves of polynomial foliations of $\mathbb{C}^2$

Based on a joint work with Yu. Kudryashov

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Outline

1. Motivation and problem statement
2. The main tool: monodromy at infinity
3. Using monodromy at infinity
4. A leaf with many handles
5. Leaves with infinitely many handles
6. Bounded limit cycles

Class $\mathcal{A}_n$

Notation

Denote by $\mathcal{A}_n$ the set of foliations of $\mathbb{C}^2$ given by

$$\dot{x} = p(x, y),$$
$$\dot{y} = q(x, y),$$

where p, q are polynomials, $\deg p \leq n, \deg q \leq n$.

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Limit and identical cycles

- Let c be a non-trivial closed loop on a leaf.
- Let $[c]$ be the free homotopy class of c.

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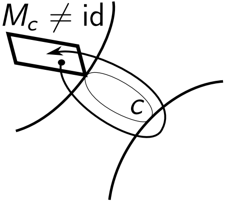

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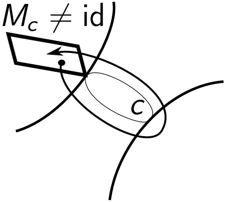

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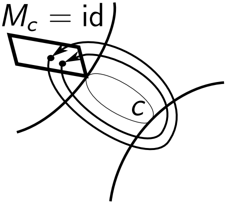

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Properties of a generic foliation $\mathcal{F} \in \mathcal{A}_n$

For a generic foliation $\mathcal{F} \in \mathcal{A}_n$,

- all leaves are dense in $\mathbb{C}^2$ (Khudai–Verenov, 1962); (Ilyashenko, 78); (Scherbakov, 1984)
- there are infinitely many independent limit cycles (Ilyashenko, 1978); (Scherbakov, Ortiz–Bobadilla, Rosales–Gonzalez, 1998);
- $\mathcal{F}$ is rigid, i.e., topological equivalence implies affine equivalence (Ilyashenko, 1978), (Scherbakov, 1984), (Nakai, 1994).

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A set of limit cycles of a foliation is called *homologically independent* if for any leaf L all the cycles located on this leaf are linearly independent in $H_1(L)$.

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Anosov conjecture

Conjecture (D.Anosov)

For a generic foliation $\mathcal{F} \in \mathcal{A}_n$,

- *countably many of its leaves are topological cylinders, others are topological discs.*
- *different limit cycles are located on different leaves.*
- *there are no identical cycles.*

Theorem (T. Firsova, 2006; T. Golenishcheva–Kutuzova, 2006)

For a foliation from some residual subset of the space of analytic foliations of $\mathbb{C}^2$, countably many of its leaves are topological cylinders, others are topological discs.

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How rare are foliations $\mathcal{F} \in \mathcal{A}_n$ with a leaf L such that $\dim H_1(L) > 1$? such that $\dim H_1(L) = \infty$?

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Main theorems

Theorem (NG, Yu. Kudryashov)

In a dense subset of $\mathcal{A}_n$, any foliation has a leaf with at least $\frac{(n+1)(n+2)}{2} - 4$ handles.

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Let $\mathcal{A}_n^{sym}$ be the subspace of $\mathcal{A}_n$, $n \geq 2$, given by

$$p(x,y) = -p(x,-y), \quad q(x,y) = q(x,-y).$$

For a foliation $\mathcal{F}$ from some open dense subset of $\mathcal{A}_n^{sym}$, all leaves of $\mathcal{F}$ (except for a finite set of algebraic leaves) have infinite genus.

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Extension to $\mathbb{C}P^2$

Change of coordinates

$$u = \frac{1}{x}; \quad v = \frac{y}{x}; \quad d\tau = -u^{n-1}dt$$

$$\begin{cases} \dot{u} = u^{n+1} p\left(\frac{1}{u}, \frac{v}{u}\right) & =: u\tilde{p}(u,v) \\ \dot{v} = vu^n p\left(\frac{1}{u}, \frac{v}{u}\right) - u^n q\left(\frac{1}{u}, \frac{v}{u}\right) & =: h(u,v). \end{cases}$$

The leaf at infinity

Let $\{ a_1, \dots, a_{n+1} \}$ be the roots of $h(0,v)$. Generically, $a_i \neq a_j$. $(0, a_j)$ are singularities of the extended foliation, and $L_\infty = \{ u = 0 \} \setminus \{ (0, a_j) \mid 1 \le j \le n+1 \}$ is its leaf.

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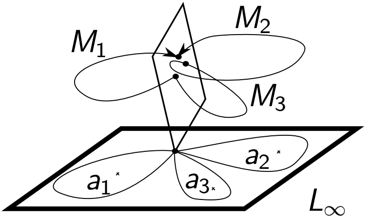
Monodromy group at infinity



Choose $O \in L_\infty$ and take loops $\gamma_j \subset L_\infty$ around a_j starting from O.
The *monodromy (pseudo)group at infinity* is generated by the monodromy maps $M_j := M_{\gamma_j}$ along γ_j.

Limit cycles correspond to isolated fixed points of monodromy maps.

Generic monodromy groups and generic foliations

Generic pseudogroup in $(\mathbb{C}, 0)$	**Generic foliation from $\mathcal{A}_n$**
• Orbits are dense in $(\mathbb{C}, 0)$	• Leaves are dense in $\mathbb{C}P^2$
• Infinite number of isolated fixed points	• Infinite number of independent limit cycles
• Rigidity	• Rigidity

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Approximating a linear map

Genericity assumptions

- $|M_1'(0)| \neq 1$, hence M_1 is linearizable;
- $\langle M_1'(0), M_2'(0) \rangle$ is dense in $\mathbb{C}^*$.

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Approximating $z \mapsto \tau z$ in the Schwartz chart for M_1

- Choose k, l such that $(M_1^k \circ M_2^l)'(0) \approx \tau$.
- Then $M_1^{-N} \circ (M_1^k \circ M_2^l) \circ M_1^N(z)$ uniformly tends to $(M_1^k \circ M_2^l)'(0)z \approx \tau z$ as $N \to \infty$.

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Under above genericity assumptions, all orbits of the monodromy pseudogroup are dense in some neighborhood of the origin.

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How to construct limit cycles

Genericity assumptions

- $\langle M_1'(0), M_2'(0)\rangle$ is dense in $\mathbb{C}^*$.
- $M_1 \circ M_2 \neq M_2 \circ M_1$.

One limit cycle

- z_0 is a fixed point of $z \mapsto \frac{z_0}{M_2(z_0)} M_2(z)$.
- Choose $M \in \langle M_1, M_2\rangle$ close to $\times \frac{z_0}{M_2(z_0)}$.
- Then $M \circ M_2$ has an isolated fixed point near z_0.
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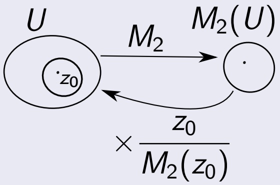

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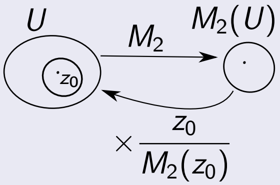

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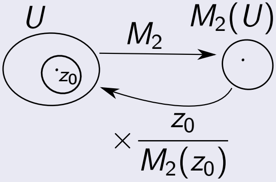

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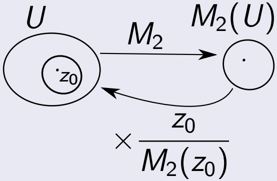

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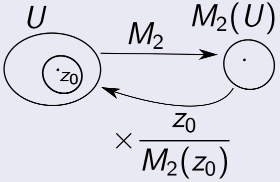

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- The homological independence is not trivial.

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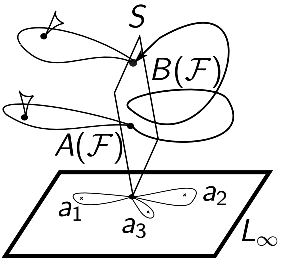
on the density of separatrix connections



Lemma (D. Volk; NG, Yu. Kudryashov)

Given a rigid foliation $\mathcal{F}$,

- *a neighborhood $\mathcal{F} \in U \subset \mathcal{A}_n$;*
- *two holomorphic functions $A, B \colon U \to S$;*

there exists $\gamma \colon S^1 \to L_\infty$ such that $M_\gamma(A(\mathcal{F})) = B(\mathcal{F})$ defines a codimension-one analytic submanifold in U.

Here $\mathcal{M}$ is such that

- $M_1'(0)$ and $M_2'(0)$ are constants on $\mathcal{M}$;
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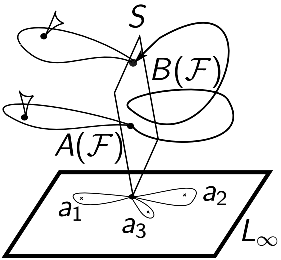

Lemma (D. Volk; NG, Yu. Kudryashov)

Given a rigid foliation $\mathcal{F}$,

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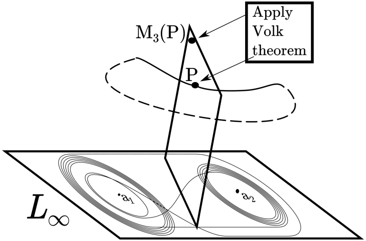
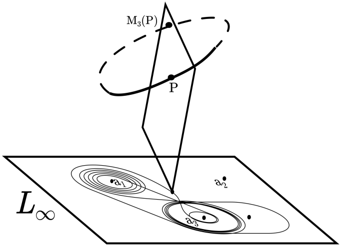
One handle

- Take one limit cycle corresponding to $M_1^{-N} \circ (M_1^k M_2^l) \circ M_1^N \circ M_2(z)$.
- Use the refined Volk's Theorem to obtain another one in a submanifold of codimension one.





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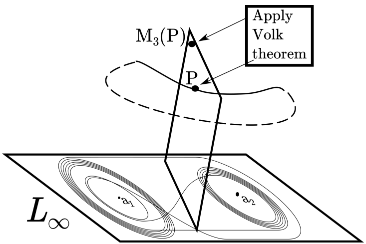

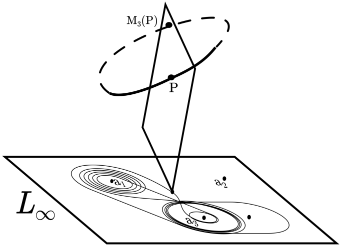

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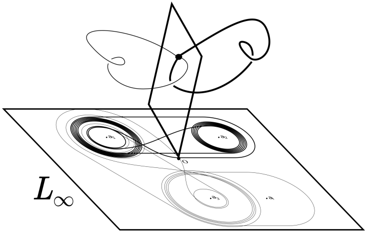

- Prove that two cycles intersect transversally at one point.

Result

In $\mathcal{M}_1 \subset \mathcal{A}_n$ with $\operatorname{codim} \mathcal{M}_1 = 1$, any $\mathcal{F} \in \mathcal{M}_1$ has one handle.

Many handles

- In $\mathcal{M}_1 \subset \mathcal{A}_n$ with $\operatorname{codim} \mathcal{M}_1 = 1$, we have one handle.
- Repeat construction inside $\mathcal{M}_1$.
- We get $\mathcal{M}_2$, $\operatorname{codim} \mathcal{M}_2 = 2$: each foliation $\mathcal{F} \in \mathcal{M}_2$ has 2 handles on different leaves, etc.;

- Use the refined Volk's Theorem to take all handles into one leaf.

The codimension is $2g - 1$, so $g \approx \frac{1}{2} \dim \mathcal{A}_n = \frac{(n+1)(n+2)}{2}$.

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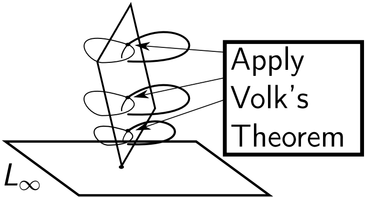

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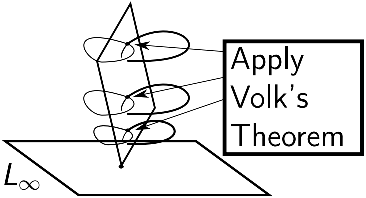

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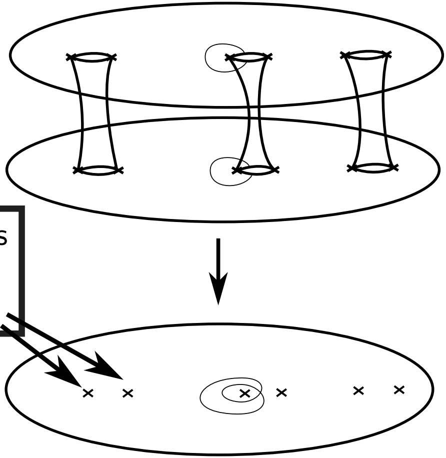
Outline

1. Motivation and problem statement
2. The main tool: monodromy at infinity
3. Using monodromy at infinity
4. A leaf with many handles
5. Leaves with infinitely many handles
6. Bounded limit cycles

Infinite genus for $p(x,y) = -p(x,-y), q(x,y) = q(x,-y)$



Intersections with $y = 0$

a leaf of $\mathcal{F}$

$\Big\downarrow (x, y^2)$

a leaf of a well-defined foliation

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Infinitely many limit cycles

Theorem (Yu.Ilyashenko, 1978)

For $n \geq 2$, each foliation $\mathcal{F}$ from some full-measure subset of $\mathcal{A}_n$ possesses infinitely many homologically independent limit cycles.

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Advantages of our approach

- The cycles are uniformly bounded.
- Subsequently, the construction survives under perturbations in $\mathcal{B}_{n+1}$.
- We estimate multipliers of the cycles instead of $\oint x\,dy - y\,dx$, and this is much simpler.

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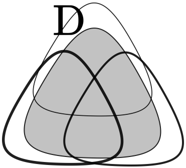

- Find a domain $0 \notin D$ and monodromy maps f_j such that
  - f_j contract in D;
  - the images of D under f_j cover D.

- Any long composition of f_j has a fixed point which produces a limit cycle.
- Multipliers tend to 0.
- Fixed points are dense in D.

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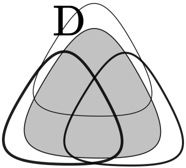

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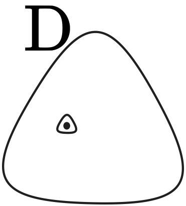

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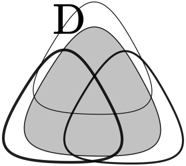

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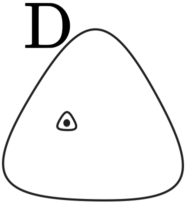

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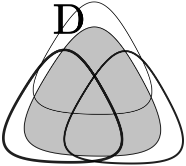

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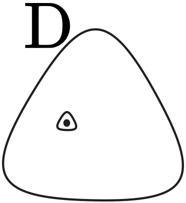

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Homological independence

Observation

If homologically dependent cycles are simple and disjoint, the dependence is of the form $c_{i_1} \pm \cdots \pm c_{i_k} = 0$ in $H_1(L)$.

Thus

- $\left(\oint_{c_{i_1}} \pm \cdots \pm \oint_{c_{i_k}}\right)(x\,dy - y\,dx) = 0$;
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