# Specification and thermodynamical properties for semigroups actions

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# Topological pressure for individual dynamics

## Classical results

$M$ compact metric space
$f : M \to M$ continuous
$\varphi : M \to \mathbb{R}$ continuous potential
The topological pressure

$$P_{\text{top}}(f,\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log\{\sup_{E_n} \sum_{x\in E_n} e^{S_n\varphi(x)}\}$$

- measures 'weighted complexity' on the space of orbits
- satisfies a *variational principle*

$$P_{\text{top}}(f,\varphi) = \sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \varphi\, d\mu\}$$

- (in some cases) measures exponential growth rate of weighted periodic points

$$P_{\text{top}}(f,\varphi) = \limsup_{n\to\infty} \frac{1}{n} \log \Big( \sum_{f^n(x)=x} e^{S_n\varphi(x)} \Big)$$

# Thermodynamical formalism for individual dynamics

## Classical results

[Adler, Konheim, McAndrew 65'] Definition of topological entropy

[Ruelle 68'] The pressure function $\beta \mapsto P_{\text{top}}(f, \varphi + \beta\psi)$ is analytic

[Bowen 71'] Specification $\Rightarrow$ Positive entropy

[Ruelle 73' Walters 75'] Variational principle for continuous maps

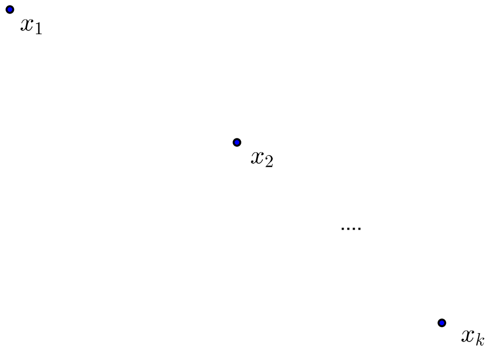
[Parry 64' Bowen 71', 74'] Specification & expansiveness
$\Rightarrow$ $\exists!$ equilibrium state $\mu_\varphi$ for every Hölder potential $\varphi$, obtained as weak$^*$-limit of

$$\frac{1}{Z_n} \sum_{x \in Per_n(f)} e^{S_n\varphi(x)} \delta_x \quad \text{with } Z_n = \sum_{x \in Per_n(f)} e^{S_n\varphi(x)}$$

and

$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log \sharp Per_n(f)$$

# Specification for individual dynamics



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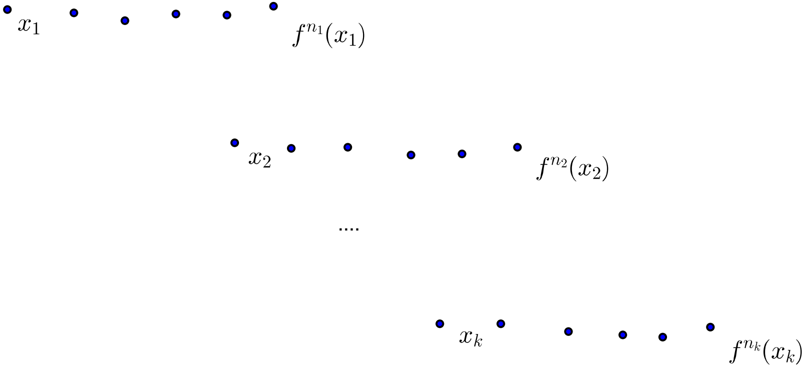

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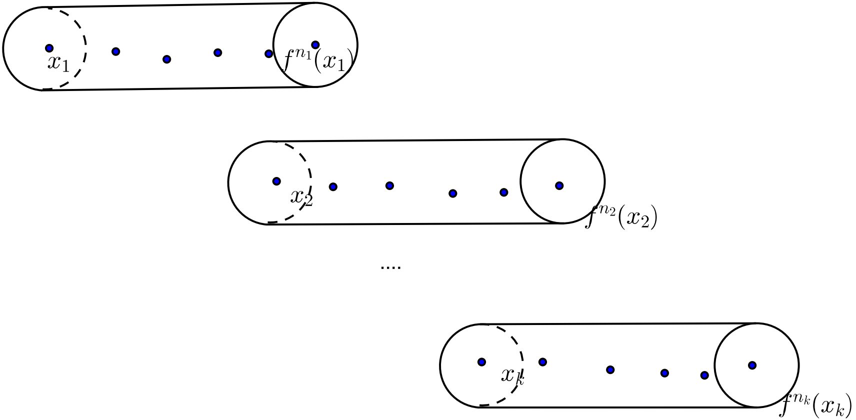

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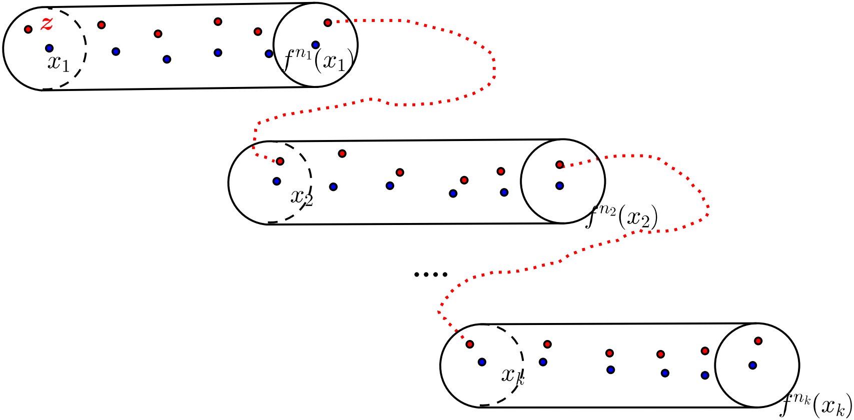
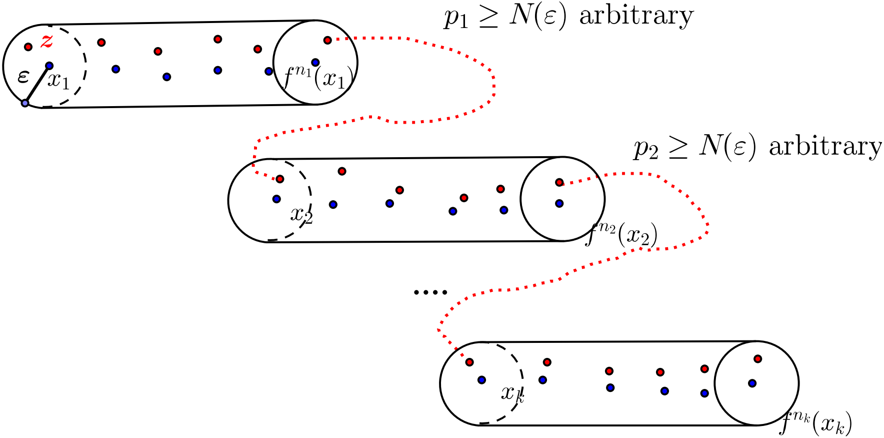


# Specification for individual dynamics



## Specification for individual dynamics: precise definition

A continuous map $f : X \to X$ satisfies the specification property if for any $\delta > 0$ there exists an integer $p(\delta) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \ldots, x_k$, and any sequence of positive integers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ with $p_i \geq p(\delta)$ there exists a point $x$ in $X$ such that

$$d\Big(f^j(x), f^j(x_1)\Big) \leq \delta, \quad \forall\, 0 \leq j \leq n_1$$

and

$$d\Big(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\Big) \leq \delta$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

# More general dynamical systems

Motivated by the number of different applications the following classes of dynamical systems have been intensively studied:

1. Non-autonomous / sequential dynamical systems
2. Iterated function systems (IFS)
3. Group and semigroup actions

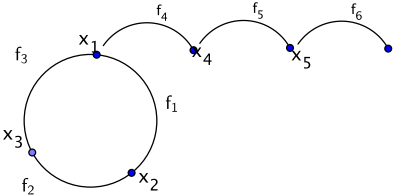
# Sequential dynamical systems

Non-autonomous (or sequential) dynamical systems $\mathcal{F} = (f_k)_{k \geq 1}$

$$F_n = f_n \circ \cdots \circ f_2 \circ f_1 \quad \text{for } n \geq 1$$

*Some difficulties* include:

- non-stationarity (no common invariant measures!)
- omega-limit sets are not necessarily invariant sets
- 'periodic points' defined by truncating dynamics




*'Topological & probabilistic complexity'*

# Finitely generated (semi)groups

$(G, \circ)$ finitely generated (semi)group

$G_1 = \{id, g_1, g_2, \ldots, g_m\}$ generators & $G = \bigcup_{n\in\mathbb{N}_0} G_n$

$\underline{g} \in G_n$ if and only if $\underline{g} = g_{i_n} \ldots g_{i_2} g_{i_1}$ with $g_{i_j} \in G_1$

(concatenations of at most $n$ elements of $G_1$)

| $(G, \circ)$ is a group | $(G, \circ)$ is a semigroup |
|---|---|
| generators $G_1 = \{id, g_1^{\pm}, g_2^{\pm}, \ldots, g_m^{\pm}\}$ | generators $G_1 = \{id, g_1, g_2, \ldots, g_m\}$ |
| $(G_n)_{n\in\mathbb{N}}$ increasing family in $G$ | $(G_n)_{n\in\mathbb{N}}$ may be non-increasing |
| $d_G(h, g) := \lvert h^{-1}g \rvert$ distance | no *natural* distance |

**Notation:** $G_1^* = G_1 \setminus \{id\}$ and

$G_n^* = \{\underline{g} = g_{i_n} \ldots g_{i_2} g_{i_1}$ with $g_{i_j} \in G_1^*\}$

## Continuous semigroup actions

We say that $T : G \times X \to X$ is a continuous semigroup action on a topological space $X$ if:

1. For every $g \in G$ the map $g \equiv T_g : X \to X$ is continuous
2. $(gh)x = g(hx)$ for every $g, h \in G$ and $x \in X$

The orbit of $x \in X$ is the set $\mathcal{O}_T(x) = \{gx \colon g \in G\}$.

$x \in X$ is 'periodic point' (period $n$) if $\underline{g}_n(x) = x$ for some $\underline{g}_n \in G_n$
$Per(G) = \bigcup_{n \geq 1} Per(G_n)$ set of periodic orbits.

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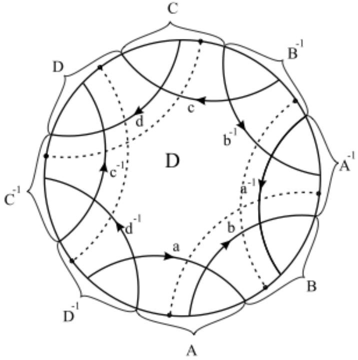
## Questions:

i. Are there *natural* notions of complexity?

ii. Can it be computed using periodic points/loops?

iii. Does local complexity propagate?

## Motivational example: geodesics and moving billiards table

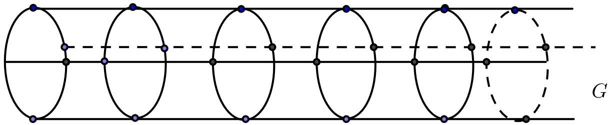



$f : \mathbb{S}^1 \to \mathbb{S}^1$ be smooth expanding map (Bowen-Series map)

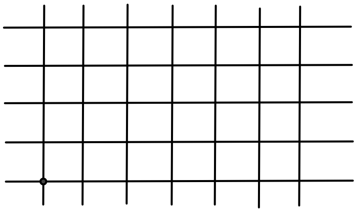
$R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ rotation angle $\alpha$

$G$ semigroup generated by $G_1 = \{id, f, R_\alpha\}$

# Coding: the semigroups $G$ and $T(G) \leq C(X,X)$



$G \simeq \mathbb{Z}_+ \times \mathbb{Z}_4$

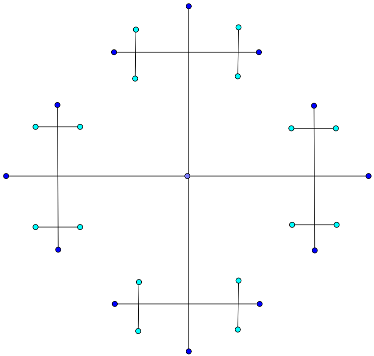


$G \simeq \mathbb{Z}_+^2$

Bijection $\mathbb{Z}_+ \times \mathbb{Z}_4 \mapsto \langle g_1(x) = R_{\frac{\pi}{4}}(x), g_2(x) = 4x(mod 1)\rangle$

Non-injective $\mathbb{Z}_+^2 \mapsto \langle g_1(x) = 2x(mod 1), g_2(x) = 4x(mod 1)\rangle$

# Coding: the semigroups $G$ and $T(G) \leq C(X, X)$



Bijection $F_2$ (free group) $\mapsto \langle g_1, g_2 \rangle$ Anosov diffeos $g_2 \notin Z(g_1)$

# Topological pressure for (semi)group actions

Some (different) notions and contributions:

[Ruelle 73']
[Ghys, Langevin, Walczak 88']
[Friedland 95']
[Bufetov 99']
[Lind, Schmidt 02']
[Bis 08', 13' ]
[Ma, Wu 11']
[Miles, Ward 11']

Some of these notions require abelianity or amenability

# Different flavours

[Ruelle 73'] $\mathbb{Z}^d$-expansive actions with (very strong) specification

2.1. **Definitions.** Let $\delta > 0$; $E \subset \Omega$ is $(\delta, \Lambda)$-separated if $(x, y \in E$, and $d(mx, my) < \delta$ for all $m \in \Lambda) \Rightarrow (x = y)$. Let $\varphi \in \mathcal{C}(\Omega)$. Given $\delta > 0$ and a finite $\Lambda \subset \mathbf{Z}^\nu$, or given $a = (a_1, \ldots, a_\nu)$ we introduce the *partition functions*

$$Z(\varphi, \delta, \Lambda) = \max_E \sum_{x \in E} \exp \sum_{m \in \Lambda} \varphi(mx) \tag{2.1}$$

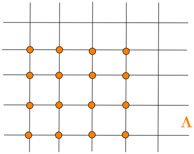
where the max is taken over all $(\delta, \Lambda)$-separated sets, or

$$Z(\varphi, a) = \sum_{x \in \Pi_a} \exp \sum_{m \in \Lambda(a)} \varphi(mx). \tag{2.2}$$

We write

$$P(\varphi, \delta, \Lambda) = (1/|\Lambda|) \log Z(\varphi, \delta, \Lambda), \tag{2.3}$$

$$P(\varphi, a) = (1/|\Lambda(a)|) \log Z(\varphi, a). \tag{2.4}$$



$\Lambda \subset \mathbb{Z}^d$

2.2. **Theorem.** *If $0 < \delta < \delta^*$, the following limits exist:*

$$\lim_{\Lambda \uparrow \infty} P(\varphi, \delta, \Lambda) = P(\varphi), \tag{2.5}$$

$$\lim_{a \to \infty} P(\varphi, a) = P(\varphi), \tag{2.6}$$

*and define a finite-valued convex function $P$ on $\mathcal{C}(X)$. Furthermore*

$$|P(\varphi) - P(\psi)| \le \|\varphi - \psi\| \tag{2.7}$$

# Different flavours

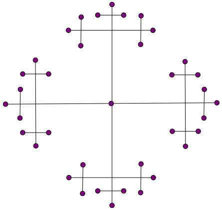
[Ghys, Langevin, Walczak 88'] Entropy for pseudo-groups and foliations

Soit $(X, d)$ un espace métrique et $\mathscr{H}_1$ une collection finie d'homéomorphismes entre des ouverts de $X$. On supposera que $\mathscr{H}_1$ contient $\mathrm{id}_X$ et que si $f \in \mathscr{H}_1$ alors $f^{-1} \in \mathscr{H}_1$. Soit $\mathscr{H}$ le pseudo-groupe d'homéomorphismes locaux engendré par $\mathscr{H}_1$. Dans cette section, nous définissons l'entropie de $\mathscr{H}$ par rapport au système de générateurs $\mathscr{H}_1$.

Si $f$ et $g$ sont dans $\mathscr{H}_1$, on n'a pas nécessairement $\mathrm{Image}(g) \subset \mathrm{Domaine}(f)$ mais nous conviendrons d'appeler composé de $f$ et de $g$ l'application $f \circ g$ définie sur $g^{-1}(\mathrm{Domaine}(f))$ (éventuellement vide). Notons $\mathscr{H}_n$ la collection des composés d'au plus $n$ éléments de $\mathscr{H}_1$ ($n \in \mathbf{N}$). Si $\varepsilon$ est un réel strictement positif et si $x, x'$ sont deux points de $X$, on dira que $x$ et $x'$ sont $(n, \varepsilon)$-séparés s'il existe un élément $f$ de $\mathscr{H}_n$ dont le domaine contient $x$ et $x'$ et tel que $d(f(x), f(x')) \geqslant \varepsilon$. Une partie $A$ de $X$ est dite $(n, \varepsilon)$-séparée si ses éléments sont $(n, \varepsilon)$-séparés deux à deux. Notons $N(\mathscr{H}, \mathscr{H}_1, n, \varepsilon)$ le cardinal maximum (éventuellement $+\infty$) des parties $A$ qui sont $(n, \varepsilon)$-séparées. On

$$h(\mathscr{H}, \mathscr{H}_1, \varepsilon) = \lim_{n \to +\infty} \sup \frac{1}{n} \operatorname{Log} N(\mathscr{H}, \mathscr{H}_1, n,$$

$$h(\mathscr{H}, \mathscr{H}_1) = \lim_{\varepsilon \to 0^+} h(\mathscr{H}, \mathscr{H}_1, \varepsilon).$$



# Different flavours

## [Bufetov 99'] Entropy free semigroup actions

Let $w \in F_m^+$, $w = w_1 w_2 ... w_k$, where $w_i = 0, 1, ..., m-1$ for all $i = 1, ..., k$. Let $f_w = f_{w_1} f_{w_2} ... f_{w_k}$. Obviously, $f_{ww'} = f_w f_{w'}$.

To each $w \in F_m^+$, we assign a metric $d_w$ on $X$ by setting

$$d_w(x_1, x_2) = \max_{w' \leq w} d(f_{w'}(x_1), f_{w'}(x_2)).$$

Clearly, if $w \leq w'$, then $d_w(x_1, x_2) \leq d_{w'}(x_1, x_2)$ for all $x_1, x_2 \in X$.

A subset of $X$ is called a $(w, \varepsilon, f_0, ..., f_{m-1})$-*spanning* subset if it is a $\varepsilon$-net on $X$ with respect to the metric $d_w$. The minimum cardinality of a $(w, \varepsilon, f_0, ..., f_{m-1})$-spanning subset of $X$ is denoted by $B(w, \varepsilon, f_0, ..., f_{m-1})$.

A subset $K$ of $X$ is called a $(w, \varepsilon, f_0, ..., f_{m-1})$-separating subset if, for any $x_1, x_2 \in K$, $x_1 \neq x_2$, one has $d_w(x_1, x_2) \geq \varepsilon$. The maximum cardinality of a $(w, \varepsilon, f_0, ..., f_{m-1})$-separating subset of $X$ is denoted by $N(w, \varepsilon, f_0, ..., f_{m-1})$.

Let

$$B(n, \varepsilon, f_0, ..., f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} B(w, \varepsilon, f_0, ..., f_{m-1}),$$
$$N(n, \varepsilon, f_0, ..., f_{m-1}) = \frac{1}{m^n} \sum_{|w|=n} N(w, \varepsilon, f_0, ..., f_{m-1})$$

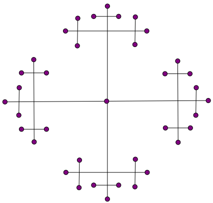
. . .

**Definition.** The topological entropy of a free semigroup action is by the formula

$$h(f_0, ..., f_{m-1}) = \lim_{\varepsilon \to 0} (\limsup_{n \to \infty}) \frac{1}{n} \log B(n, \varepsilon, f_0, ..., f_{m-1}).$$

From (2), it easily follows that

$$h(f_0, ..., f_{m-1}) = \lim_{\varepsilon \to 0} (\limsup_{n \to \infty}) \frac{1}{n} \log N(n, \varepsilon, f_0, ..., f_{m-1}).$$



# Three concepts: topological pressure & entropy points & orbital specification

## I.1 Topological pressure:

$$P_{top}((G,G_1),\varphi,E) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\left(\frac{1}{m^n}\sum_{|\underline{g}|=n}\sup_F\left\{\sum_{x\in F} e^{\sum_{i=0}^{n-1}\varphi(\underline{g}_i(x))}\right\}\right)$$

supremum over all $(\underline{g}, n, \varepsilon)$-separated sets $F = F_{\underline{g},n,\varepsilon} \subset E$

$$h_{top}((G,G_1),E) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\left(\frac{1}{m^n}\sum_{|\underline{g}|=n} s(\underline{g},n,\varepsilon)\right)$$

## I.2 Entropy:

$$h((G,G_1),E) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s(n,\varepsilon,E)$$

where $s(n,\varepsilon)$ is maximal cardinality of $(n,\varepsilon)$-separated sets in $E$.
Entropy taking the compact set $E = X$.

# Three concepts: topological pressure & entropy points & orbital specification

**Simple illustration:**

$g_1 : \mathbb{S}^1 \to \mathbb{S}^1 \qquad g_2(x) = 2x(mod1)$

$g_2 : \mathbb{S}^1 \to \mathbb{S}^1 \qquad g_2(x) = 3x(mod1)$

$g_3 : \mathbb{S}^1 \to \mathbb{S}^1 \qquad g_3(x) = 5x(mod1)$

I.1 Topological pressure:

$$h_{top}((G, G_1), \mathbb{S}^1) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{3^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, \varepsilon) \right) = \log(\frac{10}{3})$$

I.2 Entropy:

$$h((G, G_1), \mathbb{S}^1) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \mathbb{S}^1) = \log 3$$

# Three concepts: topological pressure & entropy points & orbital specification

## II.1 Entropy point

$x_0 \in X$ is an entropy point for $h_{\text{top}}((G, G_1), \cdot)$ if

$$h_{\text{top}}((G, G_1), \overline{U}) = h_{\text{top}}((G, G_1), X) \quad \text{for any open nhood } U \text{ of } x_0$$

## II.2 Entropy point

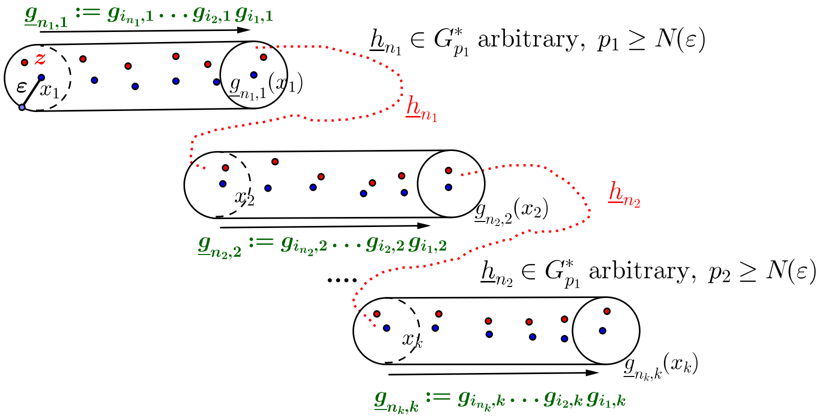
$x_0 \in X$ is an entropy point for $h((G, G_1), \cdot)$ if

$$h((G, G_1), \overline{U}) = h((G, G_1), X) \quad \text{for any open nhood } U \text{ of } x_0$$

**Rmk:** II.2 was introduced by [Bis 13'] which proved that *the set of entropy points is non-emtpy* provided $X$ is compact.

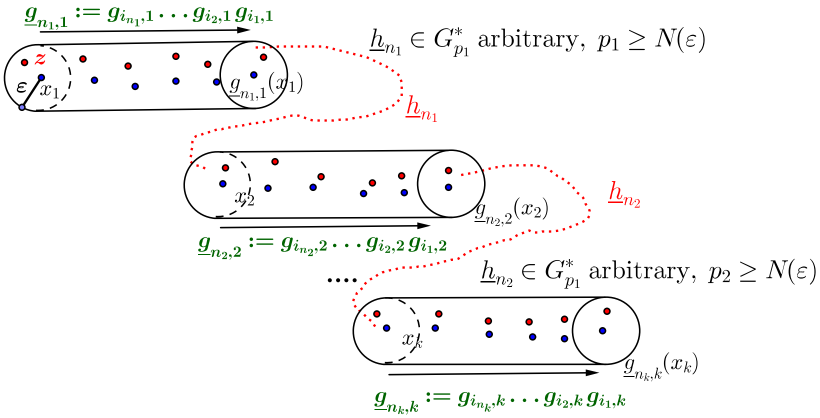
# Three concepts: topological pressure & entropy points & orbital specification

## III.1 Orbital specification



# Three concepts: topological pressure & entropy points & orbital specification

## III.1 Orbital specification

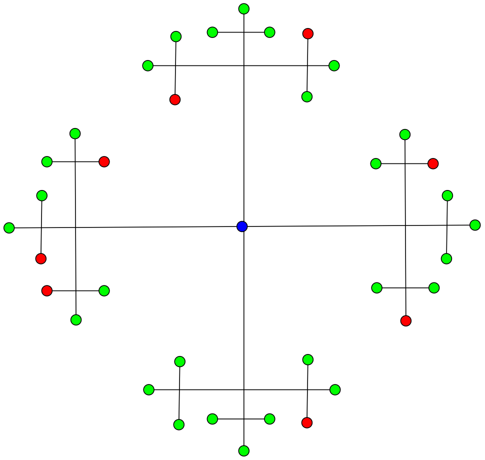



**Rmk 1:** Similar notion is studied on the space of push-forwards

**Rmk 2:** Each element in $G_1^*$ must satisfy specification

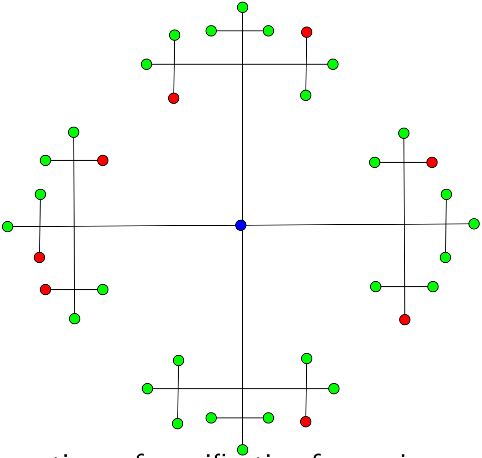
# Three concepts: topological pressure & entropy points & orbital specification

## III.2 Weak orbital specification



# Three concepts: topological pressure & entropy points & orbital specification

## III.2 Weak orbital specification



**Rmk 3:** Other notions of specification for semigroups / groups can be defined similarly (not needed for this talk!)

# Three concepts: topological pressure & entropy points & orbital specification

$T : G \times X \to X$ satisfies the weak orbital specification property if: for any $\varepsilon > 0$ there exists $p(\varepsilon) > 0$ so that for any $p \geq p(\varepsilon)$, there exists a set $\tilde{G}_p \subset G_p^*$ satisfying $\lim_{p\to\infty} \frac{\sharp \tilde{G}_p}{\sharp G_p^*} = 1$ for which: for any $h_{p_j} \in \tilde{G}_{p_j}$ with $p_j \geq p(\varepsilon)$, any points $x_1, \dots, x_k \in X$, any natural numbers $n_1, \dots, n_k$ and any concatenations $\underline{g}_{n_j,j} = g_{i_{n_j},j} \dots g_{i_2,j}\, g_{i_1,j} \in G_{n_j}$ with $1 \leq j \leq k$ there exists $x \in X$ so that $d(\underline{g}_{\ell,1}(x)\, ,\, \underline{g}_{\ell,1}(x_1)) < \varepsilon$ for every $\ell = 1 \dots n_1$ and

$$d(\, \underline{g}_{\ell,j}\, \underline{h}_{p_{j-1}}\, \cdots\, \underline{g}_{n_2,2}\, \underline{h}_{p_1}\, \underline{g}_{n_1,1}(x)\, ,\, \underline{g}_{\ell,j}(x_j)\, ) < \varepsilon$$

for every $j = 2 \dots k$ and $\ell = 1 \dots n_j$.

# Main Results

# Some results: 1. Topological pressure

**Theorem:** Let $G$ be a finitely generated semigroup with generators $G_1$. If the semigroup action induced by $G$ on the compact metric space $X$ is *strongly $\delta^*$-expansive* and the potentials $\varphi, \psi : X \to \mathbb{R}$ are continuous and satisfy the *bounded distortion property* then:

1. $P_{\text{top}}((G, G_1), \varphi + c, X) = P_{\text{top}}((G, G_1), \varphi, X) + c$, $\forall c \in \mathbb{R}$
2. $|P_{\text{top}}((G, G_1), \varphi, X) - P_{\text{top}}((G, G_1), \psi, X)| \leq \|\varphi - \psi\|$, and
3. the pressure function $t \mapsto P_{\text{top}}((G, G_1), t\varphi, X)$ is an uniform limit of differentiable maps.

Moreover, $\mathbb{R} \ni \beta \mapsto P_{\text{top}}((G, G_1), \beta\varphi, X)$ is differentiable Leb-a.e.

# Some results: 2. Positive entropy

**Theorem:** Let $G \times X \to X$ be a continuous finitely generated continuous semigroup action.

- $$\left.\begin{array}{l} \text{weak orbital specification} \\ \limsup_{p\to\infty} \dfrac{|G_p^* \setminus \tilde{G}_p|}{m^{\gamma p}} < 1, \forall 0 < \gamma < 1 \end{array}\right\} \Rightarrow h_{\text{top}}((G, G_1), X) > 0$$

- strong orbital specification $\Rightarrow h((G, G_1), X) > 0$

## Some results: 3. Local complexity

**Theorem:** Let $G \times X \to X$ be a continuous finitely generated semigroup action s.t. every element $g \in G_1$ is a local homeomorphism.

1. weak orbital specification $\Rightarrow$ every $x \in X$ is an entropy point for $h((G, G_1), \cdot)$
2. strong orbital specification $\Rightarrow$ every $x \in X$ is an entropy point for $h_{\text{top}}((G, G_1), \cdot)$

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2. strong orbital specification $\Rightarrow$ every $x \in X$ is an entropy point for $h_{\text{top}}((G, G_1), \cdot)$

**Rmk:** $h((G, G_1), \bar{U}) \leq h((G, G_1), \bar{U})$ for every $\bar{U} \subset X$
$h_{\text{top}}((G, G_1), \bar{U}) \leq h_{\text{top}}((G, G_1), X)$
$h((G, G_1), \bar{U}) \leq h((G, G_1), X)$
Although involve similar ideas, 1. and 2. are independent

## Some results: 4. Computing entropy via periodic loops

**Theorem:** Let $G$ be the semigroup generated by a set $G_1 = \{g_1, \dots, g_k\}$ of uniformly expanding maps. Then:

(a) $G$ satisfies the periodic orbital specification property,

(b) 'periodic loops' $Per(G)$ are dense in $X$, and

(c) the mean growth of periodic points is bounded from below as

$$0 < h_{top}((G, G_1), X) \leq \limsup_{n\to\infty} \frac{1}{n} \log \Big( \frac{1}{m^n} \sum_{|\underline{g}|=n} \sharp \mathrm{Fix}(\underline{g}) \Big).$$

**Rmk:** Similarly, the exponential growth rate of 'periodic loops' is larger than the entropy $h((G, G_1), X)$:

$$h((G, G_1), X) \leq \limsup_{n\to\infty} \frac{1}{n} \log \sharp Per(G_n).$$

# Two applications

1. Every $g \in G$ is an expanding map on $\mathbb{T}^n$
   - satisfy strong topological exactness
   - satisfy the orbital specification property
   - positive entropy
   - every point is an entropy point (for $h$ and $h_{\text{top}}$)
   - topological pressure is a.e. differentiable

2. $G$ generated by expanding maps ($\geq 2$) and isometries
   - the proportion of elements $\underline{h}_p \in G_p^*$ 'not suitable' for orbital specification is fastly convergent to zero as $p \to \infty$
   - satisfy the weak orbital specification property
   - every point is an entropy point (for $h$)
   - positive entropy
   - topological pressure is a.e. differentiable

Thank you!