Relativistic celestial mechanics of elastic bodies

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Construct time independent configurations in relativistic elasticity corresponding to body on circular orbit in SS

Nonrelativistic case for fluids: Lichtenstein 1933

Plan:

- structure of relativistic (time-ind.) elasticity on background $(M, g_{\mu\nu})$
- Schwarzschild case (more generally: asymptotically flat, stationary-axisymmetric vacuum spacetime)
- general scheme underlying proof: Liapunoff-Schmidt
- outlook

Relativistic elasticity

on $(M, g_{\mu\nu})$ deals with orientation-preserving maps $f : M \longrightarrow \mathcal{B}$ with $\mathcal{B}$ a domain $\subset (\mathbb{R}^3, G_{AB})$. $X^A = f^A(x^\mu)$ such that $f^A{}_{,\mu}$ has 1-dimensional, timelike kernel spanned by u^μ u^μ....4-velocity of particles making up the body. Usually $G_{AB} = \delta_{AB}$

Basic definitions: $H^{AB} = f^A{}_{,\mu} f^B{}_{,\nu}\, g^{\mu\nu}$ 'strain', energy density $\rho = \rho(H^{AB})$, number density n: $(f^\star \epsilon)_{\mu\nu\rho} = n \epsilon_{\mu\nu\rho\sigma} u^\sigma$, stress $-\sigma_{AB} = 2n \frac{\partial(\rho/n)}{\partial H^{AB}} = n \frac{\partial(e)}{\partial H^{AB}}$

action $S[f] = \int_{f^{-1}(\mathcal{B})} \rho\, dvol_g$ covariant under $Diff(M)$

$\delta S = 0$ quasilinear system of 2nd order equations for f^A

B.C. $\sigma_{AB} f^B{}_{,\mu} n^\mu |_{f^{-1}(\partial\mathcal{B})} = 0$ 'vanishing normal stress'

Next assume $(M, g_{\mu\nu})$ has timelike K.V. ξ and we restrict to configurations f^A which are time independent, i.e. $f^A{}_{,\mu}\xi^\mu = 0$. Then $S[f; g_{\mu\nu}]$ reduces to a functional $S[f; \lambda, h_{ij}]$ where $-\lambda = g(\xi, \xi)$ is a function on the quotient space N and h_{ij} the (Riemannian) metric on N.

stress tensor in space becomes $\sigma_{ij} = -2n f^A{}_{,i} f^B{}_{,j} \frac{\partial e}{\partial H^{AB}}$

field equations

$$D_j(\lambda^{\frac{1}{2}}\, \sigma_i{}^j) = ne D_i \lambda^{\frac{1}{2}} \quad , \quad \sigma_{ij} n^j|_{f^{-1}(\partial \mathcal{B})} = 0 \qquad (\star)$$

2nd-order, quasilinear, elliptic system with Neumann-type B.C.

constitutive conditions for $e = \frac{\rho}{n}$

$$e|_{H=\delta} > 0\,, \ \ \frac{\partial e}{\partial H^{AB}}|_{H=\delta} = 0$$

$$\mathring{L}_{ABCD} = \frac{\partial^2 e}{\partial H^{AB} \partial H^{CD}}|_{H=\delta} \text{ is pos.def. on } \mathbb{S}^2(\mathbb{R}^3)$$

A configuration $\mathring{f}$ is strain- and stressfree provided that $\mathring{H}^{AB} = \mathring{f}^A{}_{,i} \mathring{f}^B{}_{,j} h^{ij} = \delta^{AB}$. This clearly exists only when h_{ij} is flat. It solves field equations $(\star)$ when in addition $\lambda \equiv 1$ ('no force'). We will consider sequences $(\lambda_\epsilon, h_\epsilon)$ with $(\lambda_0 = 1, h_0 = \delta)$.

possible cases:

(1) (M, g) is Minkowski spacetime, $\xi = \partial_t + \Omega\partial_\phi$, body placed near the critical point of the (centrifugal) potential λ. This leads to the study of equilibria of an elastic body in rigid rotation. Body near-stressfree, Ω small.

(2) (M, g) is Schwarzschild spacetime with mass m, $\xi = \partial_t + \Omega\partial_\phi$, body placed near the critical point of the (gravitational + centrifugal) potential λ. Leads to the study of equilibria of elastic body near circular geodesic orbit in SS. Body near-stressfree, m, Ω small.

(3) (M, g) is asymptotically flat, stationary-axisymmetric solution of EVE's.

We will study (2)

SS case

$$g_{\mu\nu}dx^\mu dx^\nu = -(1-\tfrac{2m}{r})dt^2 + (1-\tfrac{2m}{r})^{-1}dr^2 + r^2 d\Sigma^2$$

Take K.V. $\xi = \partial_t + \Omega\,\partial_\phi$ on $N = \{2m < r < r_+\}$ with $\Omega r_+^2 = 1 - \frac{2m}{r_+}$

Clearly $\mathring{f}$ is a solution when $m = \Omega = 0$. This is unique up to composition with Euclidean motions in (N, δ_{ij}). We can assume that $\mathring{f}$ is the identity map.

Go over to material ('Lagrangian') representation by replacing f by $\Phi = f^{-1}$:

$$\nabla_A[(\lambda^{\frac{1}{2}} \circ \Phi)\sigma_i{}^A] = (eD_i\lambda^{\frac{1}{2}}) \circ \Phi\,, \qquad \sigma_i{}^A n_A|_{\partial\mathcal{B}} = 0 \quad (\star\star)$$

where $\sigma_i{}^A = \frac{\partial\hat{e}}{\partial\Phi^i{}_{,A}} = \left(H^{AB} f^C{}_{,i}\,\sigma_{BC}\right) \circ \Phi$.

By B.C., for any K.V. ξ of N,

$$\int_{\mathcal{B}}(\xi^i \circ \Phi)\nabla_A[(\lambda^{\frac{1}{2}} \circ \Phi)\sigma_i{}^A]dX = 0$$

We will scale (m, Ω) as $(m\epsilon, \Omega\epsilon^{\frac{1}{2}})$. Thus $\exists$ 6 such K.V.'s for $\epsilon = 0$.

l.h.s. of $(\star\star)$

$$\partial_A\delta\sigma_i{}^A + O(\epsilon) + O((\delta\Phi)^2), \quad \text{B.C.} \quad \delta\sigma_i{}^A n_A|_{\partial\mathcal{B}} + O(\epsilon) +$$

$O((\delta\Phi)^2)$, where $\delta\sigma_i{}^A = 2\delta^{AE}\delta^{BF}\delta^C{}_i\delta^D{}_j\mathring{L}_{CEDF}\partial_B\delta\Phi$

r.h. side of $(\star\star)$

$\epsilon[(\partial_i V) \circ \mathrm{id} + O(\epsilon) + O(\partial\Phi)]$, where $V = \frac{m}{r} + \frac{\Omega^2 r^2 \sin^2\Theta}{2}$

Map on functions $\delta\Phi^i$ on $\mathcal{B}$ given by

$\delta\Phi \mapsto (L_i = \partial_A \delta\sigma_i{}^A, l_i = \delta\sigma_i{}^A n_A|_{\partial\mathcal{B}})$ has both nontrivial kernel and range.

kernel: all Euclidean Killing vectors $\xi^i \circ \mathrm{id}$. This comes from Euclidean invariance of the l.h. side at $\epsilon = 0$

range: all pairs (L_i, l_i) for which

$$\int_{\mathcal{B}} (\xi^i \circ \mathrm{id})\, L_i\, dX - \int_{\partial\mathcal{B}} (\xi^i \circ \mathrm{id})\, l_i\, dS = 0$$

(linearized-version-of) condition: total force and total torque acting on $\mathcal{B}$ be $= 0$. Thus there is a problem in directly applying the IFT.

Idea: set $\Omega^2 = \frac{m}{L^3}$ and place 'small body' at $(r = L, \Theta = \frac{\pi}{2})$ - which corresponds to a geodesic Killing orbit (since the 'force' $D_i\lambda$ is zero there).

One then tries, by rotating, translating and scaling down $\mathcal{B}$, to arrange that $\partial_i V$ is 'equilibrated' w.r. to $\mathcal{B}$, i.e. satisfy the range condition with $l_i = 0$. Use IFT on $\mathbb{E}^3$ with parameter the 'size' of $\mathcal{B}$. Crucial linearized operator is the expression

$$H_{\mathrm{id}}[\xi, \xi'] = \int_{\mathrm{id} \circ \mathcal{B}} \mathcal{L}_\xi \mathcal{L}_{\xi'} V \, dx \, ,$$

where $\mathcal{B}$ is suitably 'small' and centered at $(r = L, \Theta = \frac{\pi}{2})$. Since $\int_{\mathrm{id} \circ \mathcal{B}} \mathcal{L}_\xi V \, dx$ is zero, this is a quadratic form on the Lie algebra of $\mathbb{E}^3$. It is degenerate w.r. to $\xi = \partial_\phi$, but otherwise non-degenerate, if all moments of inertia of $\mathcal{B}$ are unequal. This is enough for the IF argument to work.

Suppose now $\mathcal{B}$ is one just constructed.

Theorem: Let the components of Φ lie in a small neighborhood of the identity in $W^{2,p}(\mathcal{B}, \mathbb{R}^3)$ with $p > 3$ (thus Φ is in $C^1(\bar{\mathcal{B}})$ with C^1 -inverse and ${\sigma_i}^A n_A|_{\partial\mathcal{B}}$ is in $W^{1-1/p,p}(\partial\mathcal{B}, \mathbb{R}^3)$). The system $(\star\star)$ has a solution Φ_ϵ for small positive ϵ with $\Phi_0 = \mathrm{id}$. This solution is unique up to ϕ-rotations.

Idea of proof (Liapunoff-Schmidt)

step 1: solve, using the infinite-dimensional IFT, a version of $(\star\star)$, which is suitably projected onto the range of the linearized operator with fixed element of the kernel of lin.op. So this solution will depend on ϵ and 6 more parameters c_α.

step 2: solve, using the finite-dimensional IFT, for this kernel element, so that the remaining 6 conditions on the r.h.side of $(\star\star)$ are satisfied ('bifurcation equation'). So this gives $c_\alpha(\epsilon)$. (Actually one c remains undetermined due to the axial symmetry.) This essentially boils down to the argument in the preparation.

Future problems

- understand degenerate cases (not all moments of inertia different)
- constructing a full 2-body solution with helical symmetry (see Beig, Schmidt 2009 for SR scalar gravity)
- solutions with prestressed bodies: hard (see Andersson, Beig, Schmidt 2014 for one static prestressed body in Newtonian gravity)

Final note: Elastic bodies in arbitrary motion were treated in Andersson, Oliynyk, Schmidt 2014

Thank you for listening!

Concerning the 'suitable projection': actually equation is written as:

$$\partial_A \mathring{\sigma}^A{}_i = \epsilon b_i[\epsilon; \Phi]$$

$$\text{with } \mathring{\sigma}^A{}_i n_A|_{\partial\mathcal{B}} = \epsilon\tau_i[\epsilon; \Phi]$$

One then solves this, with a projection on the range of the linearized operator which leaves l_i unchanged (step 1) $\rightarrow \Phi_{\epsilon,c}$

step 2: solve for $c_\beta(\epsilon)$ the equation $\int_\mathcal{B}(\xi^i_\alpha \circ \Phi_{\epsilon,c})\, b_i[\epsilon; \Phi_{\epsilon,c}]dX = 0$ with $\alpha = 1, ..6$. This can be seen to boil down to the preparatory argument.

step 3: verify that, indeed, the full equations are satisfied.