On torsion elements in a group generated by a reversible automaton

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LIAFA – UMR 7089 CNRS & Université Paris Diderot

joint works with Th. Godin, M. Picantin, and D. Savchuk

MealyM
JCJC-12-JS02-012-01

1902

Is a finitely generated group whose all elements have finite order necessarily finite ?

Burnside

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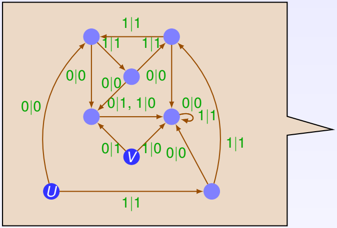
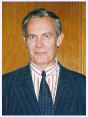
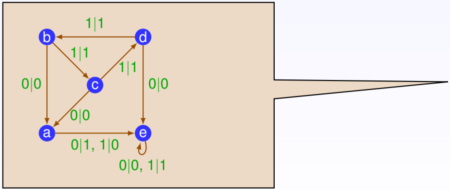
1972





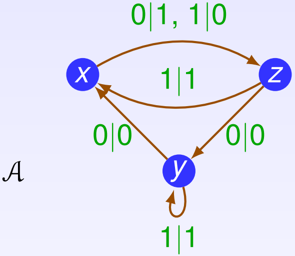
Aleshin

1980



Grigorchuk

Groups generated by automata



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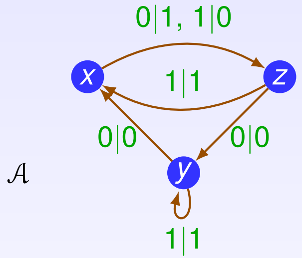
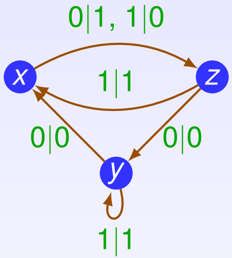


ρ_x :   01000 ↦ 11100

Groups generated by automata

$\mathcal{A}$



ρ_x : $\quad 01000 \mapsto 11100$

$\qquad \Sigma^* \to \Sigma^*$

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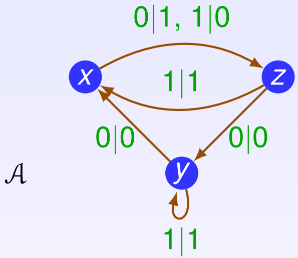

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- complete [defined all over Σ^*]

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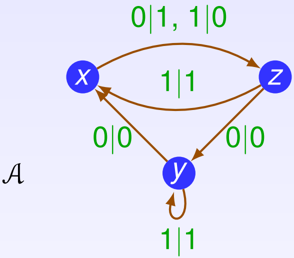

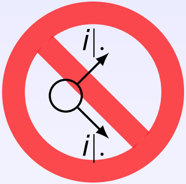

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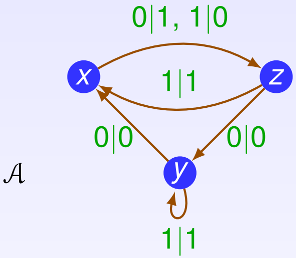

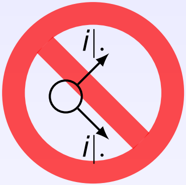

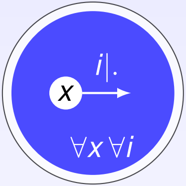

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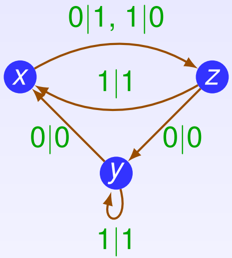
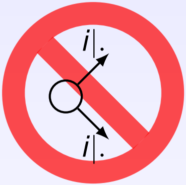
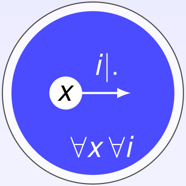


$\mathcal{A}$





$\langle \mathcal{A} \rangle_+$ semigroup

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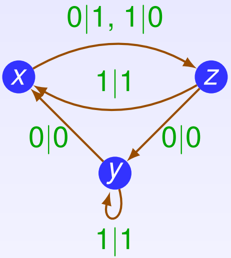
Groups generated by automata



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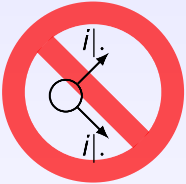

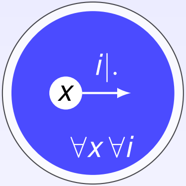

$\langle\mathcal{A}\rangle_+$ semigroup
$\langle\mathcal{A}\rangle$ group

ρ_x : $01000 \mapsto 11100$
$\Sigma^* \to \Sigma^*$

- deterministic [functional]
- complete [defined all over Σ^*]
- the states permute the alphabet [for groups]

The Burnside problem

Is a finitely generated group whose all elements have finite order necessarily finite ?

The Burnside problem

A common point between these examples ?

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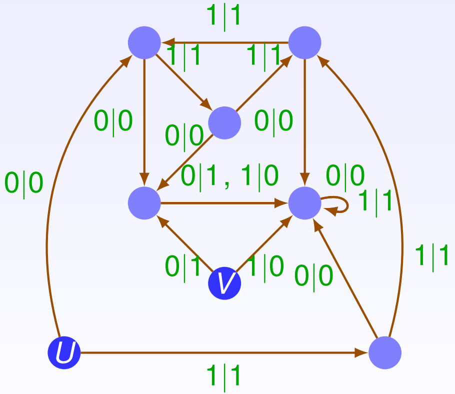

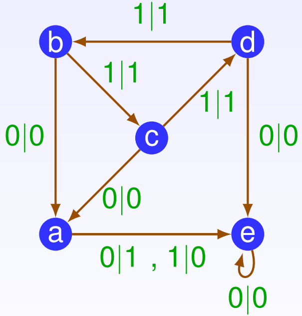

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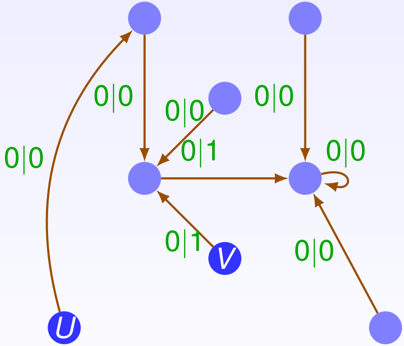

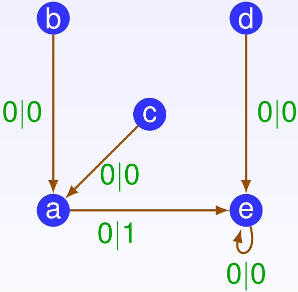

The Burnside problem and the reversibility

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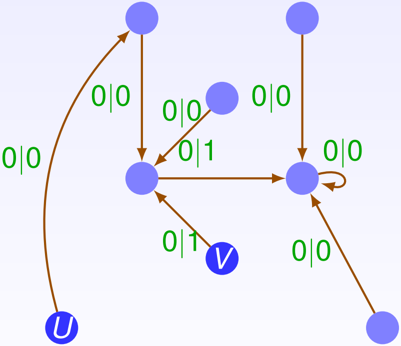

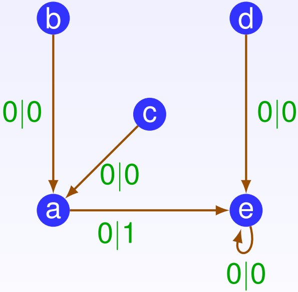

not reversible !

(letters are not permutations of the states)

The Burnside problem and the reversibility

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Can a reversible automaton generate an infinite Burnside group ?

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p prime

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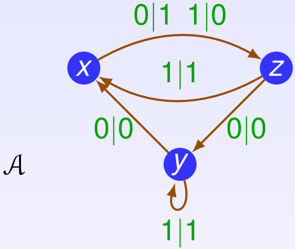

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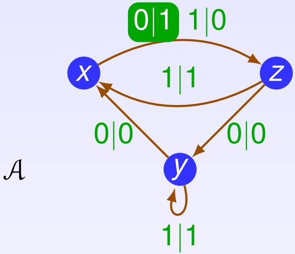
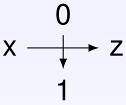
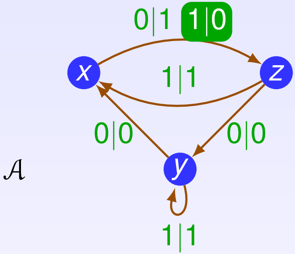
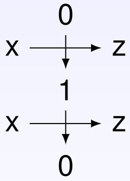




x





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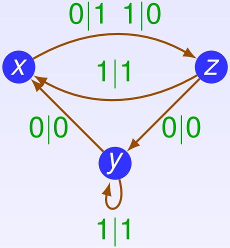

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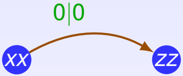

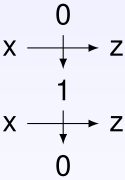

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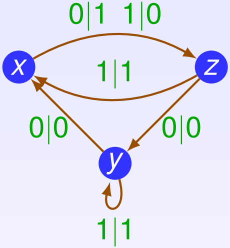

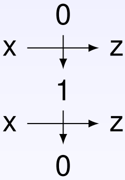

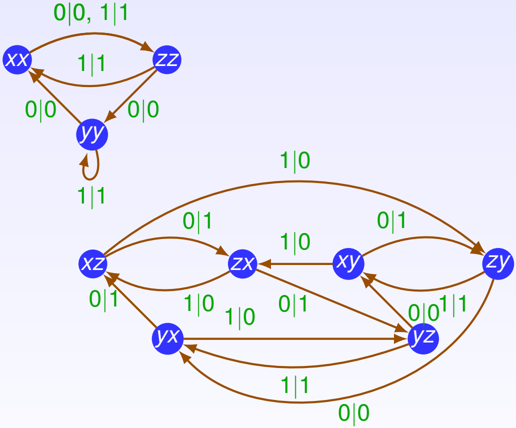

The connected components of the powers

$\mathcal{A}$ reversible (letters induce permutations of the stateset)

Finiteness problem

$\langle \mathcal{A} \rangle$ is finite $\Longleftrightarrow$ the cc of the $\mathcal{A}^n$ are bounded

Order problem

$\rho_{\mathbf{u}}$ has finite order $\Longleftrightarrow$ the cc of the $\mathbf{u}^n$ are bounded

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expressed as path properties in the orbit tree

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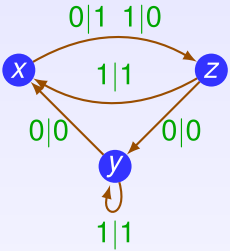

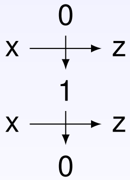

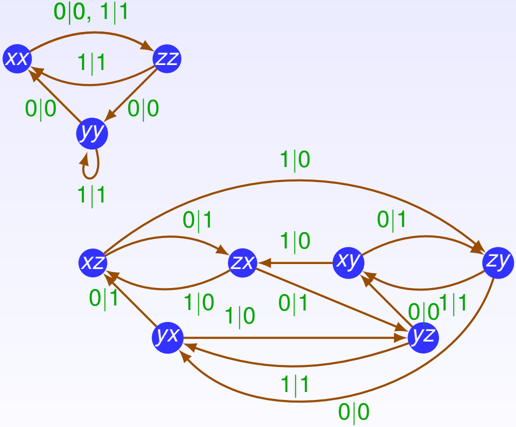

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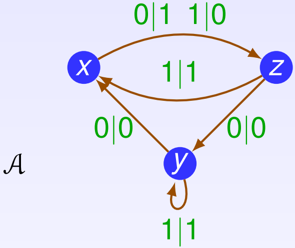
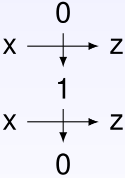
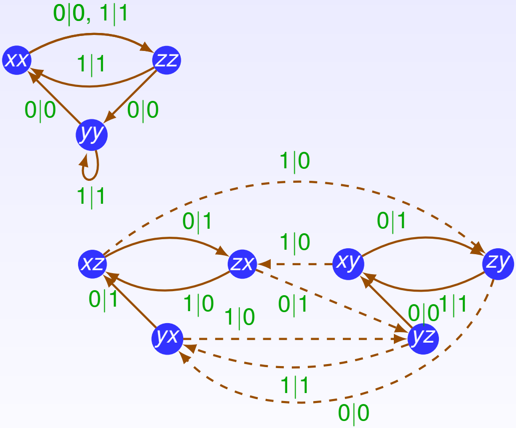
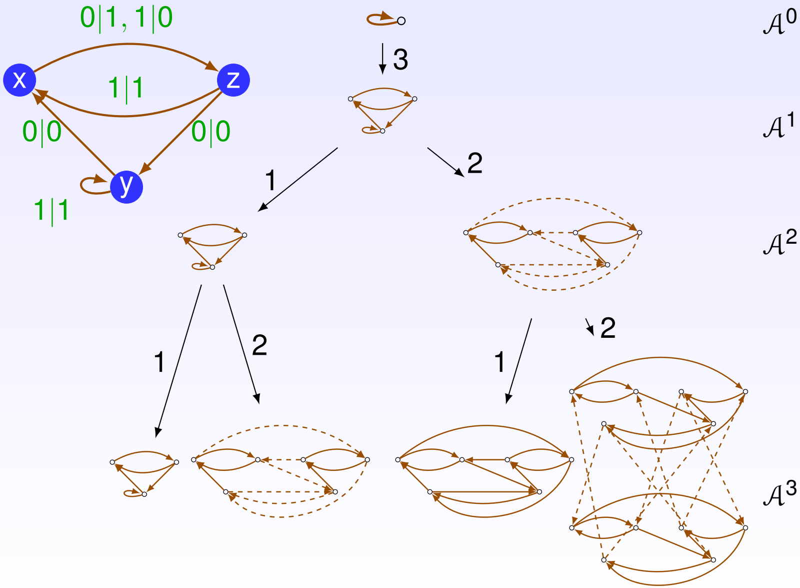






The labeled orbit tree



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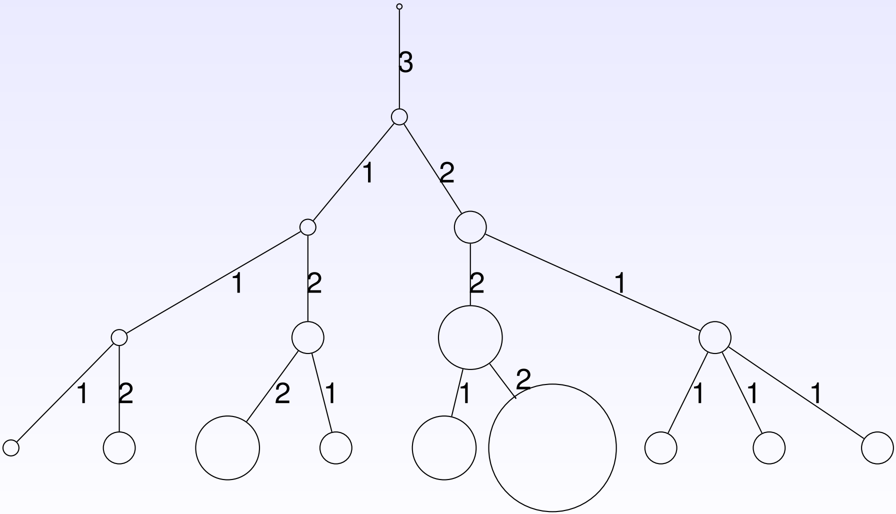

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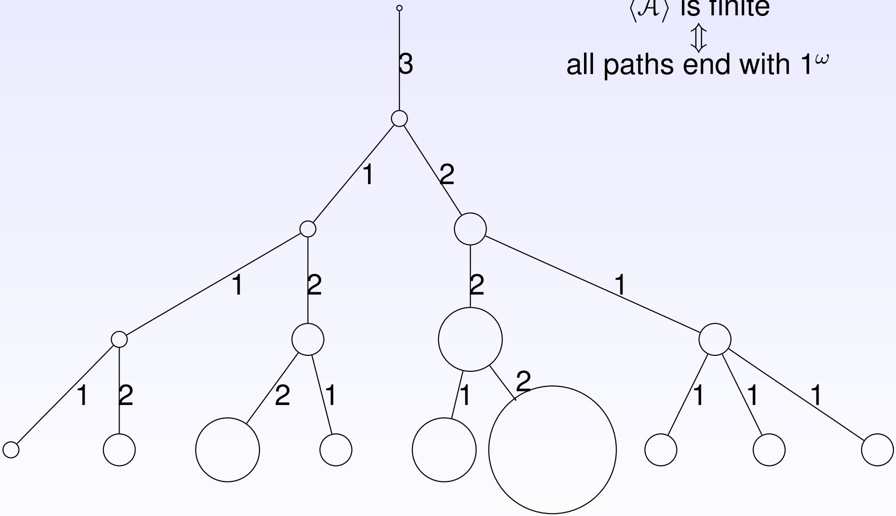

$\langle\mathcal{A}\rangle$ is finite

$\Updownarrow$

all paths end with 1^ω

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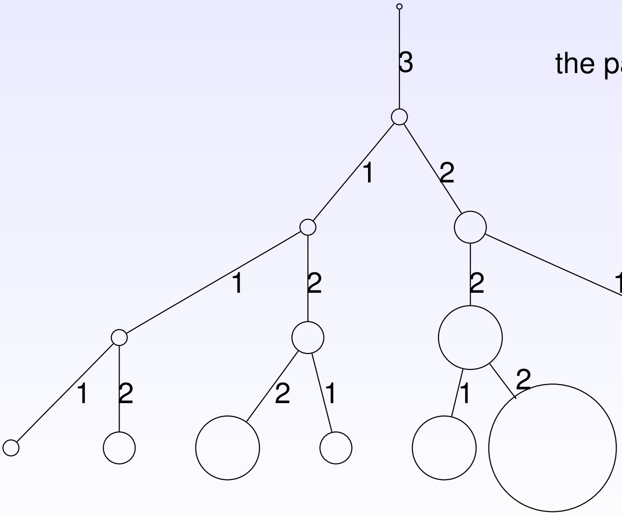


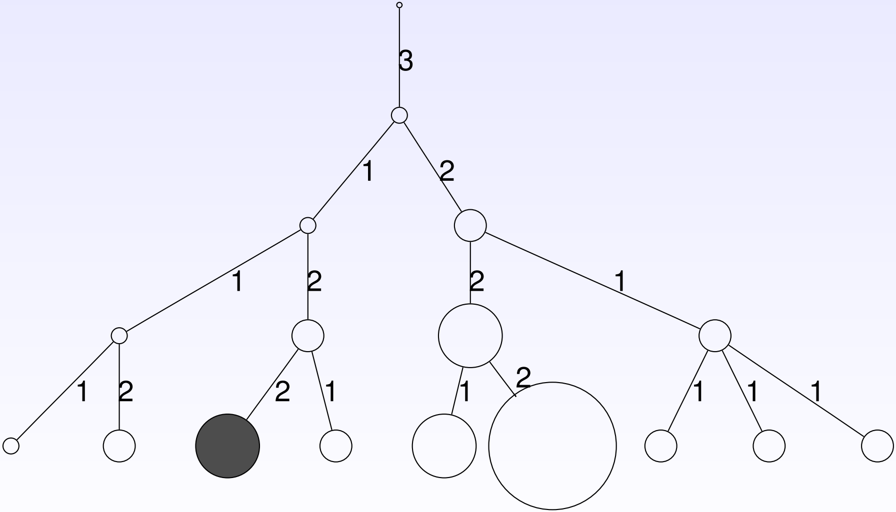
$\rho_{\mathbf{u}}$ of finite order

$\Updownarrow$

the path of $\mathbf{u}^{\omega}$ ends with 1^{ω}

The labeled orbit tree

Internal structure



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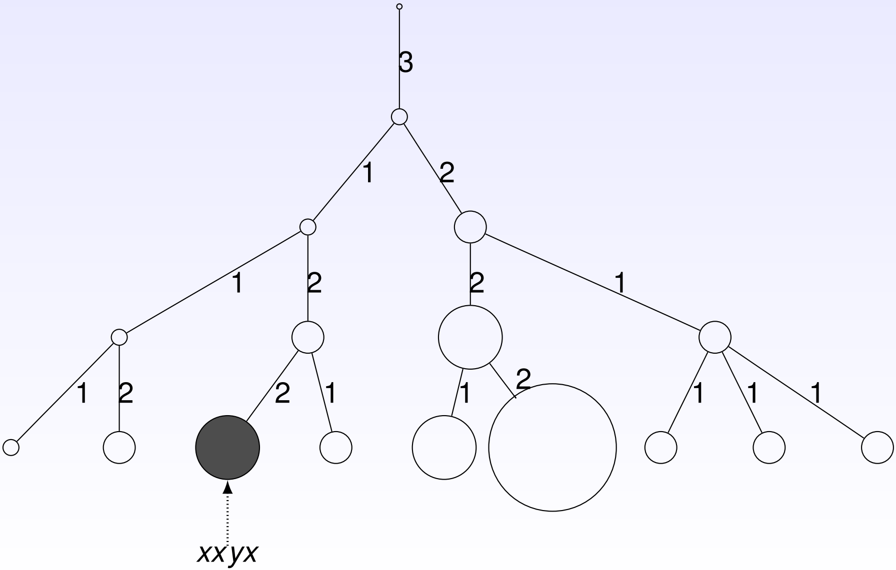

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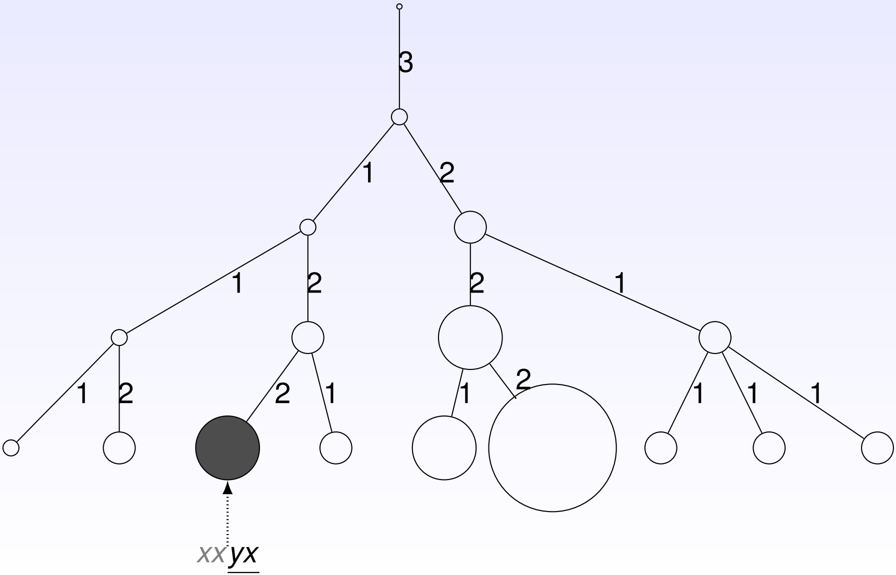

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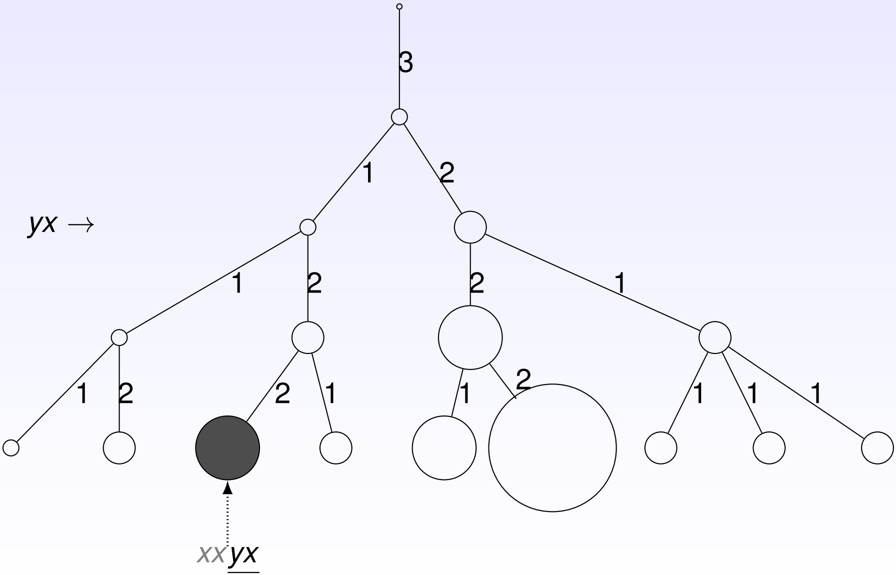

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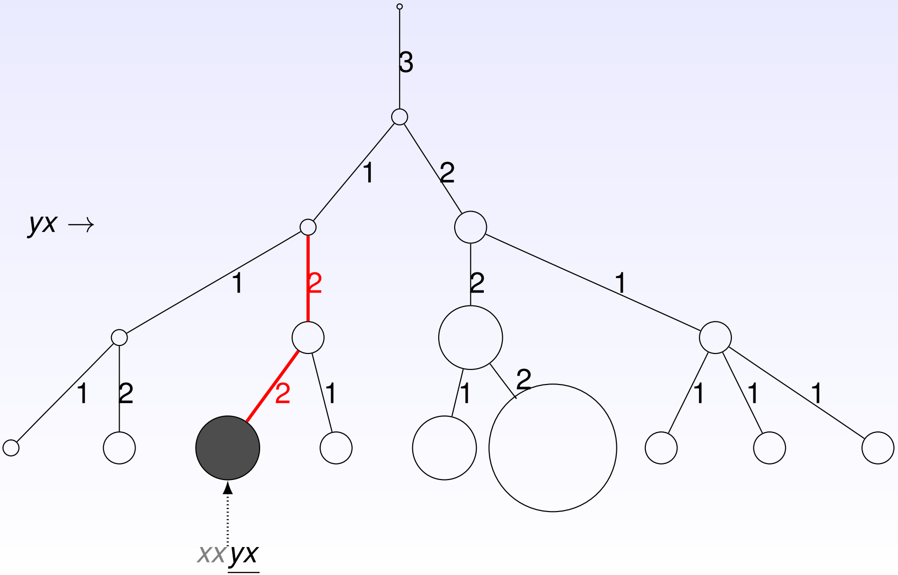

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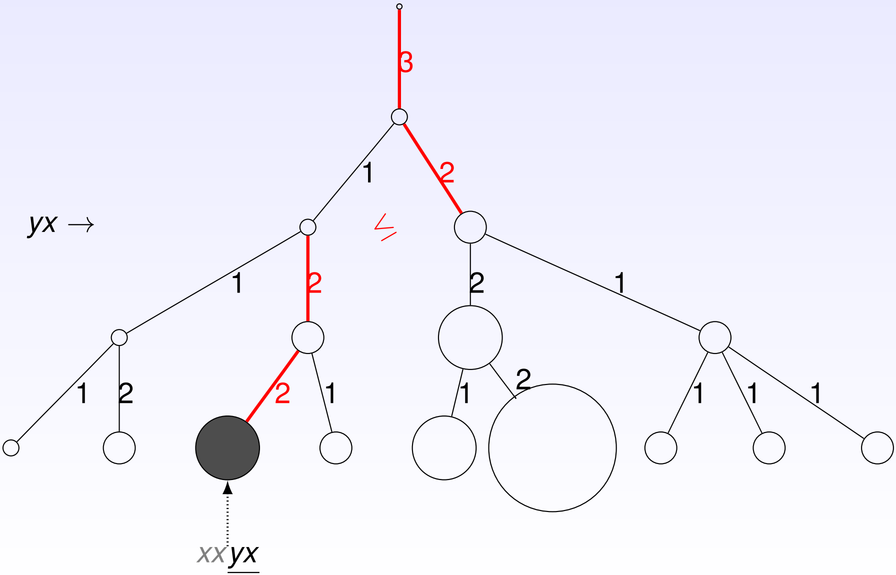

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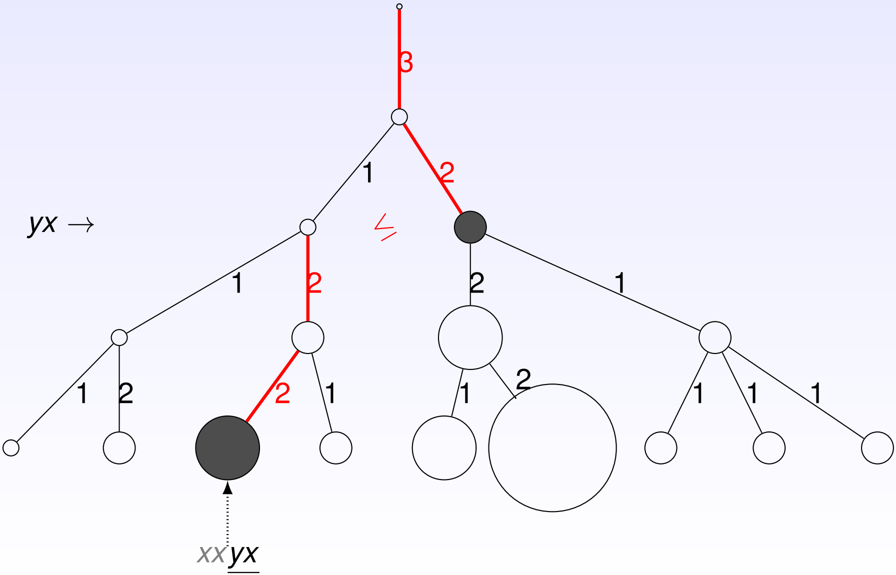

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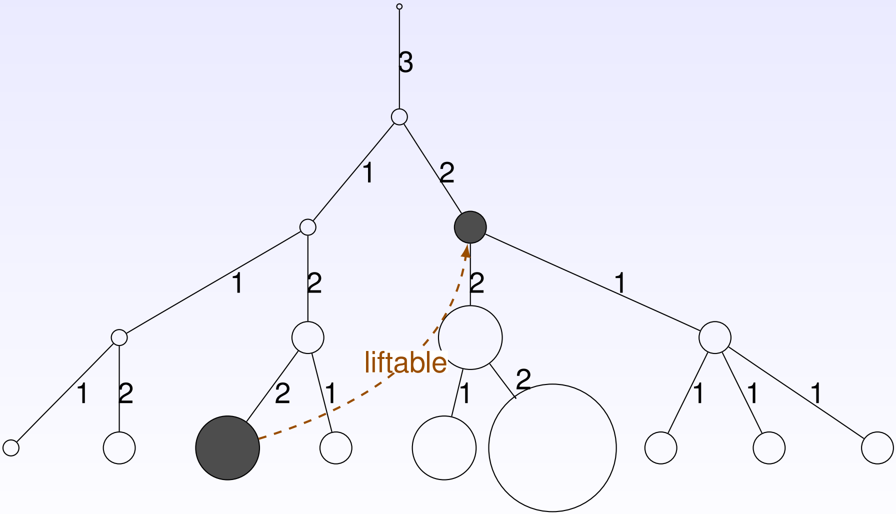
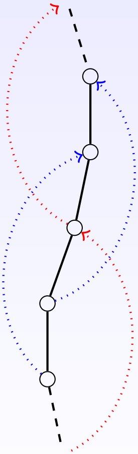


Self-liftable paths

k-self-liftable paths



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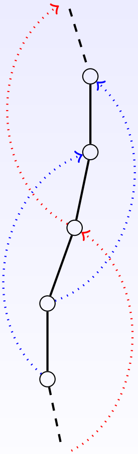

a path is self-liftable

$\Updownarrow$

it is the path of $\mathbf{u}^\omega$

First result : inv.-rev. not bireversible

[Godin K. Picantin, LATA'15]

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$\mathcal{A}$ inv.-rev. not bireversible $\Rightarrow \langle \mathcal{A} \rangle$ infinite

[Akhavi K. Lombardy Mairesse Picantin'12]

First result : inv.-rev. not bireversible

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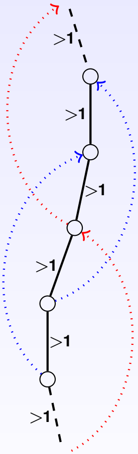

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[Akhavi K. Lombardy Mairesse Picantin'12]

in a non bireversible component

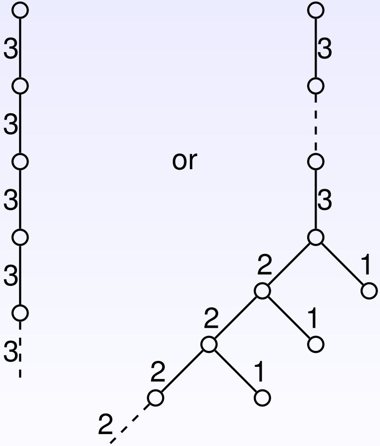
$\nexists$1 in a self-liftable path

$\Downarrow$

$\exists$ element of infinite order

Second result : 3-state connected automata

[K. Picantin Savchuk'15]



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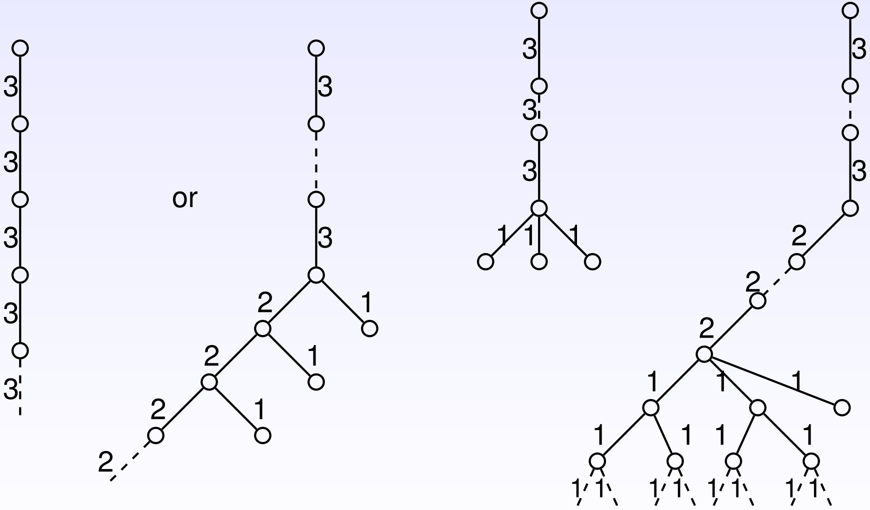
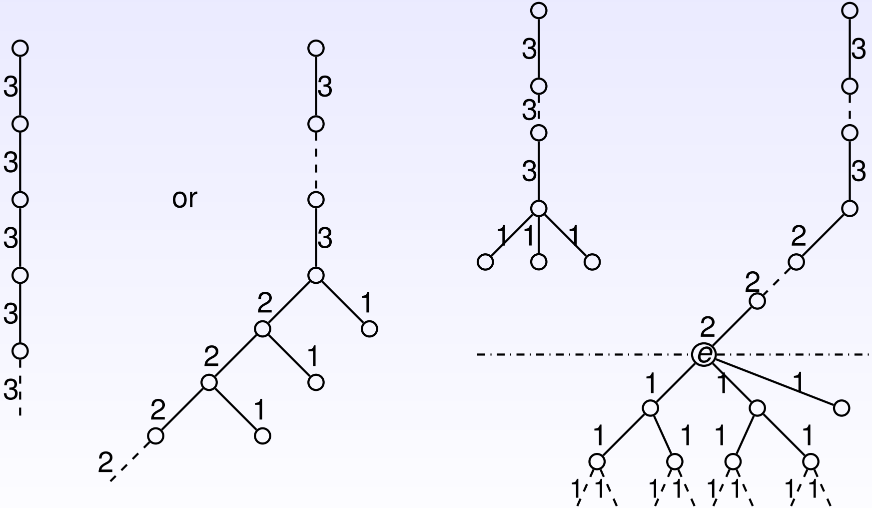


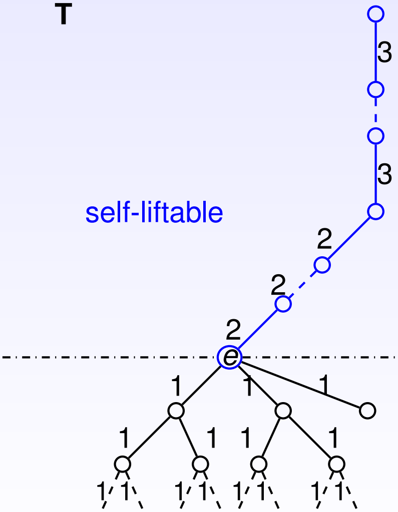
Second result : 3-state connected automata

[K. Picantin Savchuk'15]



Second result : 3-state connected automata

[K. Picantin Savchuk'15]



Second result : 3-state connected automata

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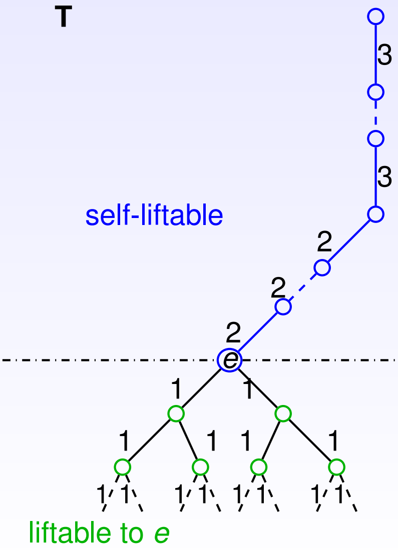
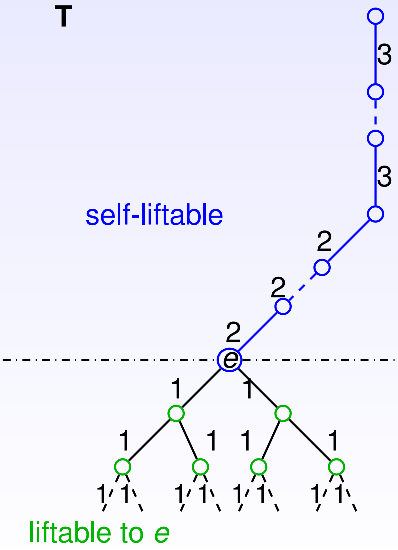


Second result : 3-state connected automata

[K. Picantin Savchuk'15]

$w \mid w^\omega \in \mathbf{T}$: bounded order

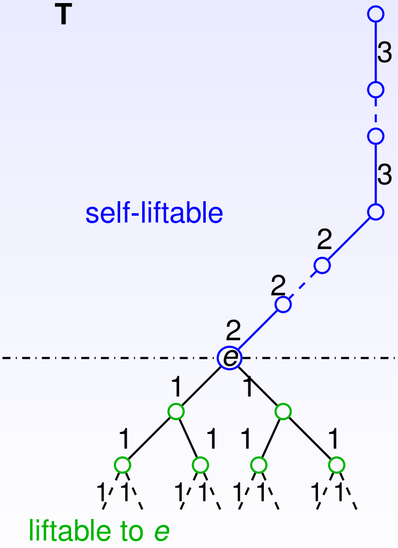


Second result : 3-state connected automata

[K. Picantin Savchuk'15]

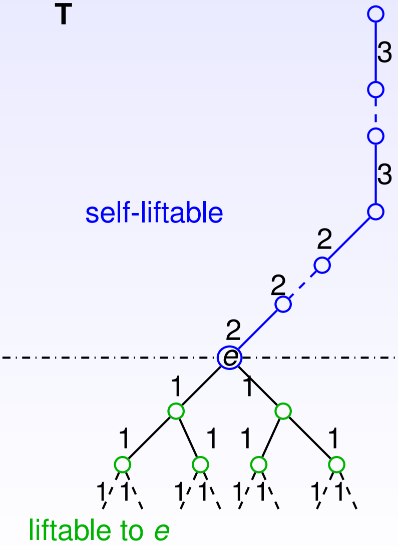
$w \mid w^\omega \in \mathbf{T}$: bounded order

$v \in Q^* : v^n \sim w$
$w \mid w^\omega \in \mathbf{T}$, bounded power



Second result : 3-state connected automata

[K. Picantin Savchuk'15]



$w \mid w^\omega \in$ **T** : bounded order

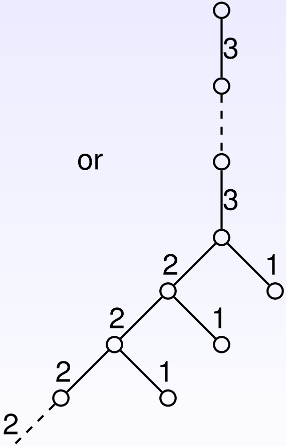
$v \in Q^* : v^n \sim w$
$w \mid w^\omega \in$ **T**, bounded power

$\Rightarrow \langle \mathcal{A} \rangle$ is finite

Second result : 3-state connected automata

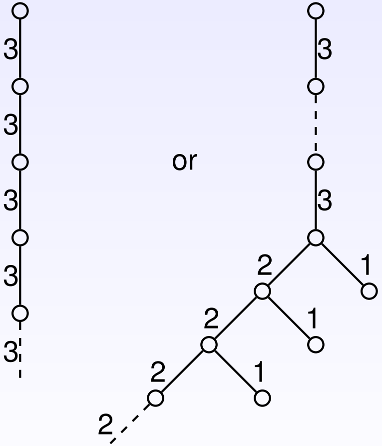
[K. Picantin Savchuk'15]

or



Second result : 3-state connected automata

[K. Picantin Savchuk'15]



$\exists\ 3^\omega$ or 3^n2^ω 1-self-liftable paths

$\Downarrow$

$\exists$ elements of infinite order

The Burnside problem and the reversibility

Natural question

Can a reversible automaton generate an infinite Burnside group ?

impossible :

- 2-state [K., STACS'13]
- connected $\textcircled{3}$-state [K. Picantin Savchuk, DLT'15] *p* prime
- non bireversible [Godin K. Picantin, LATA'15]