Intersections and sums of Gorenstein ideals

Joint work with Oana Veliche and Jerzy Weyman

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Setup

$(R, \mathfrak{m}, k)$ a commutative noetherian local ring

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$$F: \quad 0 \longrightarrow F_c \longrightarrow F_{c-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

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$$\begin{aligned} c &= \operatorname{pd}_Q \widehat{R} \\ &= \operatorname{depth} Q - \operatorname{depth}_Q \widehat{R} \\ &= \operatorname{edim} Q - \operatorname{depth} \widehat{R} \\ &= \operatorname{edim} R - \operatorname{depth} R \\ &= \operatorname{codepth} R \end{aligned}$$

Low codepth

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$$F: \quad 0 \longrightarrow Q \xrightarrow{f} Q \longrightarrow 0$$

$I = (f) \quad f \in \mathfrak{n}^2 \qquad$ *R is (abstract) hypersurface*

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Codepth 2

$$F: \quad 0 \longrightarrow Q^{n-1} \xrightarrow{\Phi} Q^n \longrightarrow Q \longrightarrow 0$$

$I = f \cdot I_{n-1}(\Phi) \qquad$ *R either*

- *(abstract) complete intersection (c.i.) (e.g.* $k[\![x,y]\!]/(x^2,y^2)$ *)*
- *Golod (e.g.* $k[\![x,y]\!]/(x^2,xy,y^2)$ *)*

Multiplicative structures

Theorem (Herzog)

If $c \leqslant 2$ then F has unique structure of differential graded (DG) algebra, i.e.

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If R Golod the algebra structure is different

Products in homology

The product on F yields product on

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Local rings by products in homology

Remark

There is always a multiplicative structure on
$\mathbf{A} = \mathrm{Tor}_*^Q(k, R) \cong \mathrm{H}(\mathrm{Koszul}^Q \otimes R) = \mathrm{H}(\mathrm{Koszul}^R)$.

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Theorem (Assmus, 1957)

R is complete intesection if and only if $\mathbf{A}$ is the exterior algebra over $\mathbf{A}_1$.

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R is Golod if and only if $\mathbf{A}$ admits a trivial Massey operation.
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Theorem (Avramov and Golod, 1971)

R is Gorenstein if and only if $\mathbf{A}$ is a Poincaré duality algebra. (The pairing $\mathbf{A_i} \times \mathbf{A_{c-i}} \to \mathbf{A_c}$ is non-degenerate and $\mathrm{rank}_k\, \mathbf{A}_c = 1$.)

Local rings by character of singularity

$$
\begin{array}{ccc}
\text{regular} & & \\
\Downarrow & & \\
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Codepth 3

Theorem (Buchsbaum and Eisenbud)

If $c \leqslant 3$ then F has a structure of an associative graded-commutative DG algebra, i.e.

$$ab = (-1)^{|a||b|} ba \qquad \text{and} \qquad a^2 = 0 \text{ for } |a| \text{ odd}$$

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Theorem (Avramov, Kustin, and Miller; Weyman)

Let $c = 3$. For fixed

$$m = \mu(I) = \operatorname{rank}_k \mathbf{A}_1 \qquad \textit{and} \qquad n = \operatorname{type} R = \operatorname{rank}_k \mathbf{A}_3$$

there are only finitely many possible structures

Possible structures

There exist bases

$$\begin{array}{lll} \mathbf{e}_1, \dots, \mathbf{e}_m & \text{for} & \mathbf{A}_1 \\ \mathbf{f}_1, \dots, \mathbf{f}_{m'} & \text{for} & \mathbf{A}_2 \quad (m' = m + n - 1) \\ \mathbf{g}_1, \dots, \mathbf{g}_n & \text{for} & \mathbf{A}_3 \end{array}$$

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Example (☺ and Veliche)

Let k be a field

$$R = \frac{k[\![x,y,z]\!]}{(xy, yz, x^3, x^2z, xz^2 - y^3, z^3)}$$

has $\operatorname{rank}_k \mathbf{A}_1 = 6$ and $\operatorname{rank}_k \mathbf{A}_3 = 2$ and it belongs to G(3). Among G rings that are not Gorenstein it is minimal w.r.t

- Dimension
- Type
- Number of generators of I
- Number of non-monomial generators of I?

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  - Inbetween: G(r) not Gorenstein G($r < m$)