

# INTERSECTIONS AND SUMS OF GORENSTEIN IDEALS

JOINT WORK WITH OANA VELICHE AND JERZY WEYMAN

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$$\begin{aligned} c &= \text{pd}_Q \widehat{R} \\ &= \text{depth } Q - \text{depth}_Q \widehat{R} \\ &= \text{edim } Q - \text{depth } \widehat{R} \\ &= \text{edim } R - \text{depth } R \\ &= \text{codepth } R \end{aligned}$$

## CODEPTH 1

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## CODEPTH 2

$$F : 0 \longrightarrow Q^{n-1} \xrightarrow{\Phi} Q^n \longrightarrow Q \longrightarrow 0$$

$I = f \cdot I_{n-1}(\Phi) \quad R \text{ either}$

- (abstract) complete intersection (c.i.) (e.g.  $k[[x, y]]/(x^2, y^2)$ )
- Golod (e.g.  $k[[x, y]]/(x^2, xy, y^2)$ )

## THEOREM (HERZOG)

*If  $c \leq 2$  then  $F$  has unique structure of differential graded (DG) algebra, i.e.*

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If  $R$  Golod the algebra structure is different

The product on  $F$  yields product on

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## REMARK

*There is always a multiplicative structure on*

$$\mathbf{A} = \mathrm{Tor}_*^Q(k, R) \cong H(\mathrm{Koszul}^Q \otimes R) = H(\mathrm{Koszul}^R).$$

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*R is complete intesection if and only if  $\mathbf{A}$  is the exterior algebra over  $\mathbf{A}_1$ .*

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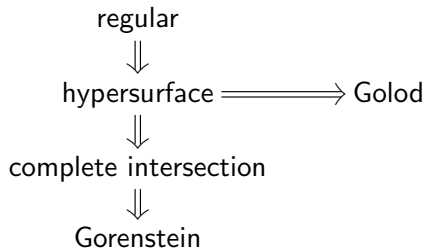
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## THEOREM (AVRAMOV AND GOLOD, 1971)

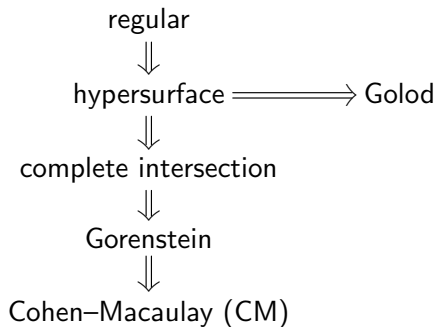
*R is Gorenstein if and only if  $\mathbf{A}$  is a Poincaré duality algebra.*  
(The pairing  $\mathbf{A}_i \times \mathbf{A}_{c-i} \rightarrow \mathbf{A}_c$  is non-degenerate and  $\mathrm{rank}_k \mathbf{A}_c = 1$ .)

# LOCAL RINGS BY CHARACTER OF SINGULARITY





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## THEOREM (BUCHSBAUM AND EISENBUD)

If  $c \leq 3$  then  $F$  has a structure of an associative graded-commutative DG algebra, i.e.

$$ab = (-1)^{|a||b|}ba \quad \text{and} \quad a^2 = 0 \quad \text{for } |a| \text{ odd}$$

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## THEOREM (AVRAMOV, KUSTIN, AND MILLER; WEYMAN)

Let  $c = 3$ . For fixed

$$m = \mu(I) = \mathrm{rank}_k \mathbf{A}_1 \quad \text{and} \quad n = \mathrm{type} R = \mathrm{rank}_k \mathbf{A}_3$$

there are only finitely many possible structures

# POSSIBLE STRUCTURES

There exist bases

$\mathbf{e}_1, \dots, \mathbf{e}_m$  for  $\mathbf{A}_1$

$\mathbf{f}_1, \dots, \mathbf{f}_{m'}$  for  $\mathbf{A}_2$  ( $m' = m + n - 1$ )

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## EXAMPLE (☺ AND VELICHE)

Let  $k$  be a field

$$R = \frac{k[[x, y, z]]}{(xy, yz, x^3, x^2z, xz^2 - y^3, z^3)}$$

has  $\text{rank}_k \mathbf{A}_1 = 6$  and  $\text{rank}_k \mathbf{A}_3 = 2$  and it belongs to  $G(3)$ .  
Among  $G$  rings that are not Gorenstein it is minimal w.r.t

- Dimension
- Type
- Number of generators of  $I$
- Number of non-monomial generators of  $I$ ?

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  - Inbetween:  $G(r)$  not Gorenstein  $G(r < m)$