Combinatorial and topological models for spaces of schedules

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Applied Algebraic Topology

TOC

Acknowledgements

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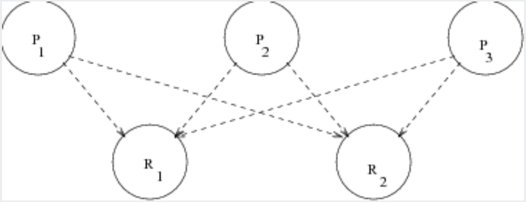
Concurrency

- In computer science, concurrency is a property of systems in which several computations are executing simultaneously and potentially interacting with each other.
- The computations may be executing on multiple cores in the same chip, in time-shared threads on the same processor, or executed on physically separated processors.
- A number of mathematical models have been developed for general concurrent computation including Petri nets and process calculi.
- Main interest here: Specific applications tuned to static program analysis – design of automated tools to test correctness etc. of a concurrent program regardless of specific timed execution.

A simple-minded approach to concurrency
Avoid access collisions

Access collisions

may occur when n processes P_i compete for m resources R_j.



Only κ (capacity) processes can be served at any given time.

Tool: Semaphores

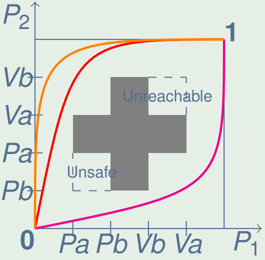
Semantics: A processor has to lock a resource and to relinquish the lock later on!
Description/abstraction: $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)
P: probeer; V: verhoog

Schedules in "progress graphs"

One semaphore on a time line

$$0 \longrightarrow P_a \longrightarrow P_b \longrightarrow V_a \longrightarrow V_b \longrightarrow 1$$

Two semaphores: The Swiss flag example



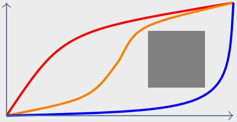
PV-diagram from
$P_1 : P_aP_bV_bV_a$
$P_2 : P_bP_aV_aV_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).
Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.
Deadlocks, unsafe and unreachable regions may occur.

Objects of study: Spaces with directed paths

First Example for impact of directedness

Directed paths in state spaces

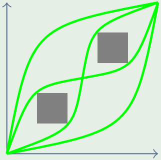


- A state space with "hole(s)"
- Paths from a start point to an end point with preferred direction: dipaths
- 1-parameter deformations of dipaths: dihomotopies

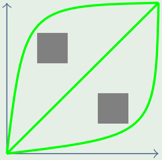
First observation

Homeomorphic state spaces may admit different types of dipaths (up to deformation):

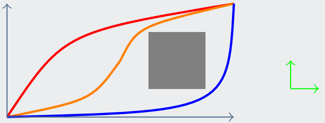
1. 4 classes:



2. 3 classes:



Directed paths in state spaces



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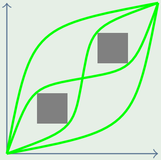
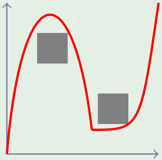


2. “forbidden” class:

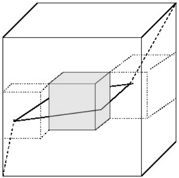


Question

Can methods from algebraic topology shed light on the space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ of directed paths – execution space – in the state space X from $\mathbf{x}_0$ to $\mathbf{x}_1$?

Problem: Symmetry breaking

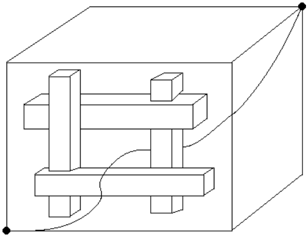
The reverse of a dipath need not be a dipath.
$\rightsquigarrow$ less structure on algebraic invariants.



"Birth and death" of dihomotopy classes

Directed topology

Loops do not tell much;
concatenation ok, cancellation not!
Replace group structure by category structures!
Example: Fundamental category $\vec{\pi}_1(X)$ – admitting a van Kampen theorem.



A dipath that is homotopic but not dihomotopic to a dipath on the boundary of the cube

- Such a deformation exists but:
- Every deformation will violate directedness.
- How to prove this?
- Remark: Need at least 3D-models for such an example!
- Space of dipaths in example $\simeq (S^1 \vee S^1) \sqcup *$.

State space $X \rightsquigarrow$ path category $\vec{P}(X)$

State spaces – three main cases of interest

- $X \subset \mathbf{R}^n$ a Euclidean cubical complex – cut out a forbidden region F consisting of hyperrectangular holes
- $X \subset \prod_i \Gamma_i$, a product of directed graphs with cubical holes (allowing branches and directed loops)
- X a directed cubical complex[a] (with directed loops): a Higher Dimensional Automaton (with labels)

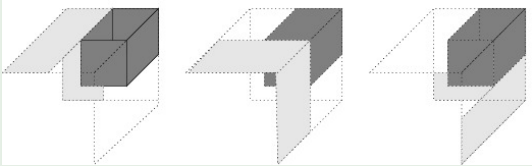
[a] as in geometric group theory

From state space X to path space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$

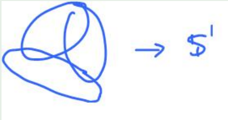
Challenge: Provide path spaces with a combinatorial (simplicial) structure

Simplicial models for spaces of dipaths

A cover of the path space associated to the “floating cube”



Cover: Dipaths through the lightgrey areas



Cover giving rise to $\partial\Delta^2 \cong S^1$

Theorem (R; 2010)

Let X be a state space consisting of a cube $\square^n$ from which l hyperrectangles are removed.
The space $\vec{P}(X)(\mathbf{0},\mathbf{1})$ of dipaths in X from bottom $\mathbf{0}$ to top $\mathbf{1}$ is homotopy equivalent to the nerve of a category $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
This category has a geometric realization as a prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subset (\partial\Delta^{n-1})^l$ – its building blocks are products of simplices.

Tool: Subspaces of state space X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

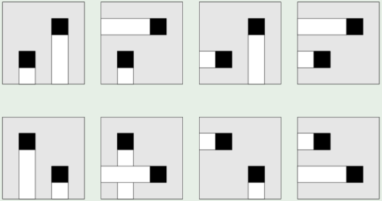
$X = \vec{I}^n \setminus F$, $F = \bigcup_{i=1}^{l} R^i$; $R^i =]\mathbf{a}^i, \mathbf{b}^i[$; $\mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition (Restricted state spaces)

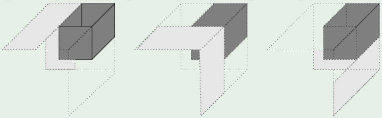
1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ – *direction j restricted at hole i*
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ – *Which directions are restricted at which hole?*

Examples: two holes in 2D – one hole in 3D (dark)

$M =$
$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$



$M =$
$[100]$ $[010]$ $[001]$



$M = [111]$: No d-path!

Covers by contractible (or empty) subspaces
Bookkeeping with binary matrices

Binary matrix posets

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices
$M^*_{l,n}$ no row vector is the zero vector –
every hole obstructed in at least one direction

Theorem (A cover by contractible subspaces)

1. $$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M^*_{l,n}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

2. *Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M^*_{l,n}$, is empty or contractible.* *Which is which? Deadlocks!*

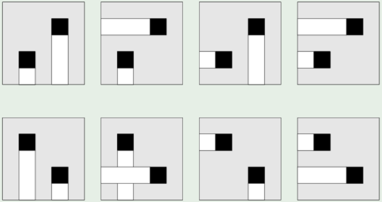
Proof.

(2) Subspaces X_M, $M \in M^*_{l,n}$ are closed under $\vee$ = l.u.b. $\square$

Combinatorics:
Poset category
$\mathcal{C}(X) \subseteq M^*_{l,n}$ consists of "alive" matrices M with $\vec{P}(X_M) \neq \emptyset$ – no deadlock!

Topology:
Prodsimplicial complex
$\mathbf{T}(X) \subseteq (\Delta^{n-1})^l$ colimit of $\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq \mathbf{T}(X)$ M alive – one simplex Δ_{m_i} for every hole.

Examples of path spaces



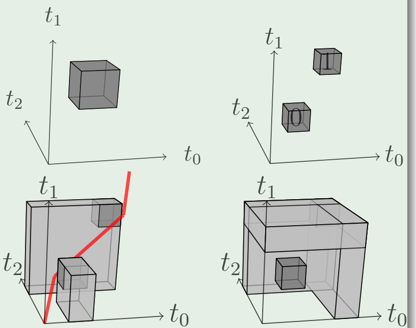
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \supset \mathcal{C}(X)$$

- $\mathbf{T}(X_1) = (\partial\Delta^1)^2 = 4*$
- $\mathbf{T}(X_2) = 3*$ – deadlock!

Further examples

State spaces, “alive” matrices and path spaces

1. $X = \vec{I}^n \setminus \vec{J}^n$
   - $\mathcal{C}(X) = M^*_{1,n} \setminus \{[1, \ldots, 1]\}$.
   - $\mathbf{T}(X) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2. $X = \vec{I}^n \setminus (\vec{J}^n_0 \cup \vec{J}^n_1)$
   - $\mathcal{C}(X) = M^*_{2,n} \setminus$ matrices with a $[1, \ldots, 1]$-row.
   - $\mathbf{T}(X) \simeq S^{n-2} \times S^{n-2}$.



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

alive dead(lock)

Theorem (A variant of the nerve lemma)

$\vec{P}(X) \simeq \Delta\mathcal{C}(X) \simeq \mathbf{T}(X)$.

allows (in principal) to calculate homology,...

Proof.

- Functors $\mathcal{D}, \mathcal{E}, \mathcal{T} : \mathcal{C}(X)^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)$,
  $\mathcal{E}(M) = \Delta_M$,
  $\mathcal{T}(M) = *$
- colim $\mathcal{D} = \vec{P}(X)$, colim $\mathcal{E} = \mathbf{T}(X)$, hocolim $\mathcal{T} = \Delta\mathcal{C}(X)$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
  hocolim $\mathcal{D} \simeq$ hocolim $\mathcal{T}^* \simeq$ hocolim $\mathcal{T} \simeq$ hocolim $\mathcal{E}$.
- Segal's projection lemma:
  hocolim $\mathcal{D} \simeq$ colim $\mathcal{D}$, hocolim $\mathcal{E} \simeq$ colim $\mathcal{E}$.

□

2. approach: Towards configuration spaces

One semaphore

Path space captured by "times" $0 < t^1 < t^2 < t^3 < t^4 < 1 \in \mathring{\Delta}_4$

n semaphores

- A directed path (n threads) is encoded by $(t_1^1, \dots, t_1^{2k_1}; \dots; t_n^1, \dots, t_n^{2k_n}) \in \prod_1^n \mathring{\Delta}_{2k_i}$
- Forbidden dipaths: Successive P_a, V_a corresponding to $t_k^{i_a}, t_k^{j_a}$.

  Capacity $n-1$: $\max_{k=1}^n t_k^{i_a} < \min_1^n t_k^{j_a}$

  Capacity κ: $\max_{1 \le k_1 < \dots < k_{\kappa+1} \le n} t_{k_j}^{i_a} < \min_{1 \le k_1 < \dots < k_{\kappa+1} \le n} t_{k_j}^{j_a}$
- The space of all forbidden dipaths A corresponds to union of a bunch of subspaces $A_{\mathbf{i},\mathbf{j}}^{\kappa+1}(a)$ of type "$\max < \min$" within $\prod_1^n \mathring{\Delta}_{2k_i}$
- Path space as configuration space: $D = \prod_1^n \mathring{\Delta}_{2k_i} \setminus A$.

Subspace arrangments

A finite set A of subspaces in affine or projective space.
Aim: To infer (topological) properties of the complement $M(A)$ from the intersection semilattice $L(A)$, partially ordered by containment.

Configuration spaces

1. $M_n(X) = \{x_1, \dots, x_n) \in X^n | \; i \neq j \Rightarrow x_i \neq x_j\}$; the complement of $A_n(X) = \bigcup_{i\neq j}\{x_i = x_j\}$ in X^n.
2. No-k-equal space $M_n^{(k)}(X)$ the complement of $A_n^{(k)}(X) = \bigcup_{1\leq i_1<\cdots<i_k\leq n}\{x_{i_1} = \cdots = x_{i_k}\}$.
3. $M_n^{(n)}(\mathbf{R}) = \mathbf{R}^n \setminus \Delta(\mathbf{R}) \simeq S^{n-2}$.
4. $M_n^{(k)}(\mathbf{R}) \subset \mathbf{R}^n$: no-$k$-equal space. Homology determined by Björner & Welker (1995); concentrated in dimensions $s(n-2)$. Cell structure and cohomology ring determined by Baryshnikov.

Path conf. spaces vs. subspace arrangements

(Dis-)similarities

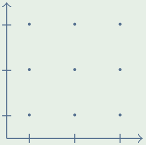
Comparison

- Path configuration space $D \subset \mathring{\Delta}_{\mathbf{k}}$ – not Euclidean (or projective).
- Complement of solutions of inequalities
- Still: Intersection semilattice matters!

A particular case: $Pa = Va$

Instantaneous use of resources. In this case:
Forbidden dipaths correspond to regions given by
equations $x^{i_1}_{j_1} = \cdots = x^{i_k}_{j_k}$.

Example: Time of access for 9 $\cdot$ obstructions, $n = k = 2$



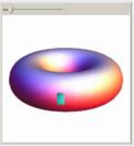
$\rightsquigarrow$ "Time space": $\mathring{\Delta}_3 \times \mathring{\Delta}_3 \setminus A \subset \mathbf{R}^6$ with
$A = \{(\mathbf{s}, \mathbf{t}) \in \mathring{\Delta}_3 \times \mathring{\Delta}_3 | \ s_i = t_j, 1 \leq i, j \leq 3\} \subset \mathbf{R}^6$.
Difficult to draw! Easy: Complement has 20 contractible components.

And in higher dimensions?

Example: Dipaths on torus – with directed loops!

Inductively, as homotopy colimits

Torus with hole (1. approach: R - Ziemiański)



Dipaths in covering of torus with hole
$\rightsquigarrow$ state space $X_n = \mathbf{R}^n \setminus (\frac{\mathbf{1}}{\mathbf{2}} + \mathbf{Z}^n)$ and of dipaths with non-negative multidegree $\mathbf{k}$ in
$Z(\mathbf{k}) := \vec{P}(X_n)(\mathbf{0}, \mathbf{k}),\ \mathbf{k} \in \mathbf{Z}^n_{\geq 0}$

Index category $\mathcal{J}(n)$

Poset category of proper non-empty subsets of $[1:n]$ with inclusions as morphisms.
Via characteristic functions isomorphic to the category of non-identical binary vectors of length n: $\boldsymbol{\varepsilon} = [\varepsilon_1, \dots \varepsilon_n] \in \mathcal{J}(n)$.
Classifying space (= nerve): $B\mathcal{J}(n) \cong \partial\Delta^{n-1} \cong S^{n-2}$.

Theorem (R-Ziemiański)

- $Z(\mathbf{k}) \simeq \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z(\mathbf{k} - \varepsilon)$.
- *Homology and cohomology can be calculated (Bousfield-Kan spectral sequence)*

Definition (Partial order on integer vectors)

$\mathbf{l} = (l_1, \ldots, l_n) \ll (m_1, \ldots, m_n) = \mathbf{m} \in \mathbf{Z}_+^n \Leftrightarrow l_j < m_j, 1 \leq j \leq n.$

Sequence complex $S(\mathbf{k})$

Vertices: $\mathbf{0} \ll \mathbf{j} \leq \mathbf{k}$

$(r-1)$-simplex: $\sigma = \mathbf{0} \ll \mathbf{j}_1 \ll \cdots \ll \mathbf{j}_r \leq \mathbf{k}$

$\partial_i \sigma$: omit $\mathbf{j}_i$.

Order complex $\Delta(\mathbf{k})$ for arrangement

Consider arrangement $A(\mathbf{k}) = \bigcup_{1 \leq i_j \leq k_j} \{x^1_{i_1} = \cdots = x^n_{i_n}\}$ within compactification $\widehat{\mathring{\Delta}}_{\mathbf{k}} \cong S^{|\mathbf{k}|}$.
Order complex $\Delta(\mathbf{k})$: poset of (non-empty) intersections.
OBS: "Unordered" intersections give rise to point $\star$ at infinity.

Experiment: Configuration spaces and wedge lemma

- Configuration space for $Z(\mathbf{k})$:
  $D(\mathbf{k}) := \mathring{\Delta}_{\mathbf{k}} \setminus A(\mathbf{k}) = \mathring{\Delta}_{k_1} \times \cdots \times \mathring{\Delta}_{k_n} \setminus A(\mathbf{k}) \subset \widehat{\mathring{\Delta}}_{\mathbf{k}} \cong S^{|\mathbf{k}|}$ with $A(\mathbf{k}) = \bigcup_{1 \le i_j \le k_j} \{x^1_{i_1} = \cdots = x^n_{i_n}\}$ within compactification.
- (Co-)homology of $\widehat{A}(\mathbf{k}) \subset \widehat{\mathring{\Delta}}_{\mathbf{k}} = S^{|\mathbf{k}|}$ using the intersection poset Q of the cover defined by $A(\mathbf{k}) \rightsquigarrow$ Alexander duality $H_*(D(\mathbf{k}))$

Application of Wedge lemma (Ziegler-Živaljević 1995)

1. $\widehat{A}(\mathbf{k}) \simeq \bigvee_{q \in Q} \Delta(Q_{<q}) * U_q$ – $\Delta(Q_{<q})$ the order complex "below q", U_q the intersection corresponding to q.
2. $q = (\mathbf{j}_1 \ll \cdots \ll \mathbf{j}_r) \in Q \Rightarrow \Delta(Q_{<q}) \simeq S^{r-2}$ and $U_q = S^{|\mathbf{k}| - r(n-1)}$.
3. q "unordered"$\Rightarrow U_q = *$ – does not contribute !
4. $\widehat{A}(\mathbf{k}) \simeq \bigvee_{q=(\mathbf{j}_1 \ll \cdots \ll \mathbf{j}_r) \in Q} S^{|\mathbf{k}| - r(n-2) - 1}$.

Schedules with capacity κ

Replace semaphores of capacity $n-1$ by semaphores with capacity κ. Schedules can be viewed as

- dipaths on κ-skeleton of $\mathbf{R}^n$ (cubified)
- elements in the complement $D^{\kappa+1}(\mathbf{k})$ of $A^{\kappa+1}(\mathbf{k}) =$
  $= \{x_{j_1}^{i_1} = \cdots = x_{j_{\kappa+1}}^{i_{\kappa+1}} \mid 1 \leq j_1 < \cdots < j_{\kappa+1} \leq n, 1 \leq i_s \leq k_{j_s}\}$
  in $\hat{\mathring{\Delta}}_{\mathbf{k}}$

Strategy

Again use wedge lemma and Alexander duality.
Relevant order complexes: Joins of order complexes of partition complexes – non-singleton parts of size at least $\kappa+1$.
These are homotopy equvalent to wedges of spheres (Björner, Welker; 1995).

Homology and cohomology of path spaces in torus skeleta

Theorem (Meshulam-R)

1. $\tilde{H}^{|\mathbf{k}|-l-1}(\widehat{A}(\mathbf{k});\mathbf{Z}) = \begin{cases} \mathbf{Z}^{\prod_{i=1}^{n}\binom{k_i}{r}} & l=(n-2)r,\ r>0 \\ 0 & \textit{otherwise} \end{cases}$
2. $\tilde{H}_l(D(\mathbf{k});\mathbf{Z}) = \begin{cases} \mathbf{Z}^{\prod_{i=1}^{n}\binom{k_i}{r}} & l=(n-2)r,\ r>0 \\ 0 & \textit{otherwise} \end{cases}$
   alternative proof of the result of R-Ziemiański
3. $H_*(D^{\kappa+1}(\mathbf{k});\mathbf{Z})$ *is concentrated in dimensions* $r(\kappa-1),\ r\in\mathbf{Z}_{\geq 0}$.
   Poincaré series can be identified.

A sketch in the special case

- An element $\mathbf{x} = (x_1, \cdots x_k) \in \mathring{\Delta}_k$ gives rise to a directed piecewise linear path $p_{\mathbf{x}} : I \to [0, k+1]$ with
$$p_{\mathbf{x}}(t) = \begin{cases} 0 & t = 0 \\ i & t = x_i \\ k+1 & t = 1 \end{cases}.$$
- An element $\underline{\mathbf{x}} = (\mathbf{x}_1, \cdots \mathbf{x}_n) \in \prod_1^n \mathring{\Delta}_{k_i} = \mathring{\Delta}_{\mathbf{k}}$ gives rise to a directed piecewise linear path $P_{\underline{\mathbf{x}}} : I \to \mathbf{R}^n,\ P_{\underline{\mathbf{x}}}(t) = (p_{\mathbf{x}_1}(t), \dots, p_{\mathbf{x}_n}(t))$ from $\mathbf{0}$ to $\mathbf{k}$.
- Only the forbidden configurations in A ((in)-equalities) correspond to dipaths through the forbidden region F (placing the V, P at integers).
- The map $\mathring{\Delta}_{\mathbf{k}} \setminus A \to \vec{P}(\prod_i [0, k_i + 1] \setminus F)(\mathbf{0}, \mathbf{k}+\mathbf{1}) : \underline{\mathbf{x}} \to P_{\underline{\mathbf{x}}}$ is a homotopy equivalence.

Can easily be generalized!

Prodsimplicial vs. configuration space model

A comparison

Dimensions

prodsimplicial model Dimension $\leq l(n-1)$, l the number of "holes" (multiplicative)

configuration space Dimension $\leq 2\sum_i k_i$ (additive)

Questions. Comments

- Can one use the wedge lemma strategy to determine the homotopy type of the complement of the configuration space – in general?
- Determine the (stable) homotopy type of the configuration space?
- Its homology? Algorithmically?
- Observe: Complicated order complexes in general!

Moment angle complexes

Introduced by Davis and Januszkiewicz (1991) in toric topology.
Many properties exhibited by Buchstaber and Panov (2000).
Here: Generalized moment-angle complexes (GMAC).

A GMAC associated to the sequence complex $S(\mathbf{k})$

- Remember $\sigma = (0 \ll \mathbf{j}_1 \ll \cdots \ll \mathbf{j}_r \leq \mathbf{k}) \in S_{r-1}(\mathbf{k})$
- $D(\sigma) = \prod_{\mathbf{j}\in\sigma} S^{n-2} \times \prod_{\mathbf{j}\notin\sigma} \star \subset \prod_{\mathbf{0}\ll\mathbf{j}\leq\mathbf{k}} S^{n-2}$
- $MA(\mathbf{k}) = \bigcup_{\sigma\in S(\mathbf{k})} D(\sigma) = \mathrm{colim}\, D(\sigma)$.
- Easy to calculate (co-)homology via a Mayer-Vietoris argument.

Question/Conjecture:Path space as GMAC

$Z(\mathbf{k}) \simeq MA(\mathbf{k})$?

Which topological spaces can arise as path spaces?

Surely there must be restrictions?

Theorem (No: K. Ziemiański; 2013)

For every finite simplicial complex X (on n vertices) there is

1. *a Euclidean cubical complex $X_E \subset \mathbf{R}^n$ such that $\vec{P}(X_E)(\mathbf{0},\mathbf{1}) \simeq X$.*
2. *a linear PV-program (n concurrent PV threads) with cubical realization Y_E such that $\vec{P}(Y_E)(\mathbf{0},\mathbf{2}) \simeq X \sqcup S^{n-2}$.*

Thanks!

Thanks

- to you, the audience
- to the organizers
- the sponsor