Free idempotent generated semigroups and the endomorphism monoid of a free G-act

Groups and idempotents

Victoria Gould

University of York

AMS-EMS-SPM meeting
Porto, 12th June 2015

Idempotents: $e = e^2$

Throughout, S is a **semigroup**.

$E = \{e \in S : e \text{ is idempotent}\} = E(S)$.

S is **idempotent generated** if

$$S = \langle E \rangle = \{e_1 \dots e_n : e_i \in E\}.$$

Howie 1966

Every semigroup embeds into some $\langle E \rangle$.

Idempotents: we love them!

Howie 1966

$\mathcal{T}_n \setminus \mathcal{S}_n$ is idempotent generated.

J.A. Erdös 1967, Laffey 1973, Dawlings 1979

$M_n(D) \setminus GL(n, D)$ is idempotent generated for any division ring D.

Putcha 2006

Conditions for the singular ideal of a reductive linear algebraic monoids to be idempotent generated.

Idempotents: everyone loves them!

Idempotent operators in analysis and mathematical physics

Topological semigroups - the Stone-Čech compactification $\beta(\mathbb{N})$

Our motivation today:

Nambooripad's theory of biordered sets

Free G-acts over a group G

The **free G-act** $F_n(G)$ on $\{x_1, \dots, x_n\}$ is given by

$$F_n(G) = Gx_1 \cup Gx_2 \cup \dots \cup Gx_n = \{gx_i : g \in G, 1 \leq i \leq n\}$$

where

$$g(hx_i) = (gh)x_i.$$

For $\alpha \in \operatorname{End} F_n(G)$ we have $(gx_i)\alpha = g(x_i\alpha)$ so we write

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ g_1 x_{1\overline{\alpha}} & g_2 x_{2\overline{\alpha}} & \dots & g_n x_{n\overline{\alpha}} \end{pmatrix} \longleftrightarrow (g_1, g_2, \dots, g_n; \overline{\alpha})$$

giving

$$\operatorname{End} F_n(G) \cong G \wr_n \mathcal{T}_n.$$

Idempotent generated semigroups: an essential example

For $\alpha \in \operatorname{End} F_n(G)$, define

$$\operatorname{rank}\alpha = |\operatorname{Im}\overline{\alpha}| = \operatorname{rank}\overline{\alpha}$$

Sing End $F_n(G)$: ideal

Sing End $F_n(G)$ is

$$\{\alpha \in \operatorname{End} F_n(G) : \operatorname{rank}\alpha < n\}$$

End $F_n(G)$

Aut $F_n(G)$: rank n

Sing End $F_n(G)$
rank $< n$

I : rank 1

Fountain and Lewin 1992

Sing End$(F_n(G))$ is idempotent generated.

A maximal subgroup in End $F_n(G)$ of rank 1

Let $n \geq 2$. Put

$$\epsilon = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & x_1 & \dots & x_1 \end{pmatrix}.$$

Then it is easy to see

$$H_\epsilon = \left\{ \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ gx_1 & gx_1 & \dots & gx_1 \end{pmatrix} : g \in G \right\} \cong G.$$

$$H_\epsilon = \epsilon E \epsilon.$$

Check:

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ gx_1 & gx_1 & \dots & gx_1 \end{pmatrix} =$$

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & x_1 & \dots & x_1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ gx_2 & x_2 & \dots & x_2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & x_1 & \dots & x_1 \end{pmatrix}.$$

Nambooripad's theory of biordered sets: $E = E(S)$ for a semigroup S

Fact E does not have to be a subsemigroup.

E is closed under **basic products** where ef is a basic product if

$$e \leq_{\mathcal{R}} f,\ f \leq_{\mathcal{R}} e,\ e \leq_{\mathcal{L}} f \text{ or } f \leq_{\mathcal{L}} e.$$

Here $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ are the pre-orders associated with Green's relations $\mathcal{R}$ and $\mathcal{L}$.

Nambooripad's theory of biordered sets

- Under basic products, E satisfies a number of axioms; if S is regular, an extra axiom holds.
- A **biordered set** is a partial algebra satisfying these axioms; if the extra one also holds it is a **regular biordered set**.

Nambooripad 1979, Easdown 1985

E is a (regular) biordered set if and only if $E = E(S)$ for a (regular) semigroup S.

Free idempotent generated semigroups

Let E be a biordered set. We can assume $E = E(S)$ for a semigroup $S = \langle E \rangle$.
Let
$$\overline{E} = \{\overline{e} : e \in E\}.$$

IG(E)

$$\mathrm{IG}(E) = \langle \overline{E} : \overline{e}\,\overline{f} = \overline{ef}, ef \text{ is a basic product} \rangle$$

- $\mathrm{IG}(E) = \langle \overline{E} \rangle$.
- $\overline{e} \mapsto e$ is an onto morphism $\mathrm{IG}(E) \to \langle E \rangle = S$.
- $E(\mathrm{IG}(E)) = \overline{E} \cong E$, as a biordered set.

In view of the above, $\mathrm{IG}(E)$ is called the **free idempotent generated** semigroup on the biordered set E.

Maximal subgroups of $\mathrm{IG}(E)$

Questions

Given an idempotent generated semigroup S, find the groups $H_{\overline{e}}$ in $\mathrm{IG}(E)$.

More particularly, find when $H_{\overline{e}} \cong H_e$, that is, e is **good**.

These questions have a long and distinguished history:

Nambooripad , Pastijn 1970s/1980s,
McElwee, Easdown, Brittenham, Margolis, Meakin 2000s,
BMM, Gray, Ruškuc, Dolinka 2010s.

We consider $\mathrm{IG}(E)$ where $E = E(\mathrm{End}\, F_n(G))$.

Who is we?

Yang Dandan

Maximal subgroups of $\mathrm{IG}(E)$ where $E = E(\mathrm{End}\, F_n(G))$

Theorem: Yang Dandan and G 2014

Let G be a group and let $n \geq 3$ and put $\epsilon = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_1 & x_1 & \dots & x_1 \end{pmatrix} \in E(\mathrm{End}\, F_n(G))$. Then ϵ is good:

$$H_{\bar{\epsilon}} \cong H_{\epsilon} \cong G.$$

Corollary: Gray and Ruškuc, 2012

Any group appears as a maximal subgroup of some $\mathrm{IG}(E)$.

End $F_n(G)$ - going up

To consider the question of whether $H_{\overline{\epsilon}} \cong H_\epsilon$ for *higher rank* ϵ, we need to work harder. Much harder.

We use the presentation of $H_{\overline{\epsilon}}$ given by Gray and Ruškuc in 2012.

Who is we now?

Igor Dolinka

End $F_n(G)$ - going up

Theorem: Dandan, Dolinka and G 2015

Let G be a group and let $n \geq 3$ and let $\epsilon \in E = E(\operatorname{End} F_n(G))$ with rank $\epsilon = r \leq n - 2$. Then ϵ is good:

$$H_{\overline{\epsilon}} \cong H_\epsilon \cong G \wr_r \mathcal{S}_r.$$

Corollary: Gray, Ruskuc 2011

If $e \in E(\mathcal{T}_n)$ and rank $e = r \leq n - 2$, then $H_{\overline{e}} \cong H_e \cong \mathcal{S}_r$.

What groups can arise as maximal subgroups of $\mathrm{IG}(E)$?

- All such groups discovered in 1970s, 80s, 90s were free.
- 2002 it was formally conjectured that all such groups were free.
- 2009 **Brittenham, Margolis and Meakin** found $\mathbb{Z} \oplus \mathbb{Z}$ as a maximal subgroup of some $\mathrm{IG}(E)$.

Gray, Ruskuc 2012

Any group can arise in this way.

When are idempotents good?

- c. 2010 **Brittenham, Margolis and Meakin** If $A \in E(M_n(D))$ and $\operatorname{rank} A = 1$, then $H_{\overline{A}} \cong H_A \cong D^*$.
- c. 2011 **Dolinka, Gray** If $A \in E(M_n(D))$ and $\operatorname{rank} A < n/3$, then $H_{\overline{A}} \cong H_A \cong \mathrm{GL}(r, D)$.
- c. 2011 **Gray, Ruskuc** If $e \in E(\mathcal{T}_n)$ and $\operatorname{rank} e = r \leq n - 2$, then $H_{\overline{e}} \cong H_e \cong \mathcal{S}_r$.

Put G to be trivial in $\operatorname{End} F_n(G)$ to obtain the result for $\mathcal{T}_n$.

- For all of these cases, the rank $n - 1$ idempotents produce free groups in $\mathrm{IG}(E)$.