On the geometry of the unit ball in a JB*-triple

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Jordan *-triple
complex vector space A together with a **triple product** i.e., a mapping

$$\{\dots\} : A \times A \times A \to A$$
$$(a, b, c) \mapsto \{a \ \ b \ \ c\}$$

- symmetric and linear in the outer variables
- conjugate linear in the middle variable
- $D(a, b) : A \to A$ linear operator

$$D(a, b)c = \{a \ \ b \ \ c\}$$

$$[D(a, b), D(c, d)] = D(\{a \ \ b \ \ c\}, d) - D(c, \{b \ \ a \ \ d\})$$

JB^*-triple is a Jordan *-triple A such that:

- A is a Banach space
- triple product is continuous
- $D(a,a)$ hermitian, with non-negative spectrum and

$$\|D(a,a)\| = \|a\|^2$$

JBW^*-triple if A is the dual of a Banach space

Examples

- $M_{n\times k}(\mathbb{C})$ is a JB*-triple
- C*-algebra is a JB*-triple for the triple product defined by

$$\{a\ \ b\ \ c\} = \frac{1}{2}(ab^*c + cb^*a) \qquad (1)$$

- W*-algebra is JBW*-triple with the triple product (1)
- The bidual of a JB*-triple is a JBW*-triple

V complex Banach space

V^* dual of V

V_1 unit ball of V

V_1^* unit ball of V^*

$\mathcal{F}_n(V_1)$ norm-closed faces of V_1

$\mathcal{F}_{w^*}(V_1^*)$ weak*-closed faces of V_1^*

$$C \subseteq V, \quad C \text{ convex}$$

$$E \subseteq C, \quad E \text{ convex}$$

E face of C *if* for $z_1, z_2 \in C$

$$\exists 0 < t < 1 \qquad tz_1 + (1-t)z_2 \in E \Rightarrow z_1, z_2 \in E$$

- $S \subseteq V_1$

$$\underbrace{S' = \{x \in V_1^* \colon x(a) = 1,\ \forall a \in S\}}_{\text{weak*-closed face of } V_1^*}$$

- $W \subseteq V_1^*$

$$W_\prime = \underbrace{\{a \in V_1 \colon x(a) = 1,\ \forall x \in W\}}_{\text{norm-closed face of } V_1}$$

- $F \subseteq V_1$ is a norm-exposed face of V_1 if

$$F = \{x\}_{\prime} \qquad \text{for some } x \in V_1^*$$

- $G \subseteq V_1$ is a weak*-exposed face of V_1^* if

$$G = \{a\}' \qquad \text{for some } a \in V_1$$

- $\mathcal{E}_n(V_1)$ norm-exposed faces of V_1

- $\mathcal{E}_{w^*}(V_1^*)$ weak*-exposed faces of V_1^*

$F \in \mathcal{F}_n(V_1)$ norm-semi-exposed $\rightarrow$ intersection of norm-exposed faces

$G \in \mathcal{F}_{w^*}(V_1)$ weak*-semi-exposed $\rightarrow$ intersection of weak*-exposed faces

$\mathcal{E}_n(V_1) \subseteq \mathcal{S}_n(V_1) \subseteq \mathcal{F}_n(V_1)$ $\qquad$ $\mathcal{E}_{w^*}(V_1^*) \subseteq \mathcal{S}_{w^*}(V_1^*) \subseteq \mathcal{F}_{w^*}(V_1^*)$

$\mathcal{S}_n(V_1), \mathcal{F}_n(V_1), \mathcal{S}_{w^*}(V_1^*), \mathcal{F}_{w^*}(V_1^*)$ together with "$\subseteq$" are complete lattices

A JB*-triple

A^* dual of A

A_1 unit ball of A

A_1^* unit ball of A^*

F norm-closed face of A_1

$\uparrow$ norm-exposed?

$\uparrow$ norm-semi-exposed?

G weak*-closed face of A_1^*

$\uparrow$ weak*-exposed?

$\uparrow$ weak*-semi-exposed?

- $u \in A^{**}$ $\quad u$ tripotent if $\{u\,u\,u\} = u$

- Peirce decomposition

$$A^{**} = A_0^{**}(u) \oplus A_1^{**}(u) \oplus A_2^{**}(u)$$

$$A_j(u) = \{a \in A : D(u,u)a = \tfrac{1}{2}ja\} \qquad \text{for} \qquad j = 0, 1, 2$$

- Peirce projections

$$P_2(u) = Q(u,u)^2 \qquad Q(u,u)a = \{u\ \ a\ \ u\}$$

$$P_1(u) = 2(D(u,u) - Q(u,u)^2) \qquad P_0(u) = \mathrm{I} - 2D(u,u) + Q(u,u)^2$$

- u, v tripotents, $\mathcal{U}(A^{**})$ set of tripotents

$$u \leq v \qquad \text{if} \qquad v - u \in A_0^{**}(u)$$

"$\leq$" partial order relation

- $\underbrace{\mathcal{U}(A^{**}) \cup \{\omega\}}_{\mathcal{U}(A^{**})\tilde{}}$ together with $\leq$ is a complete lattice

(ω greatest element)

(Edwards–Rüttimann '87)

Theorem

1. *The map*

$$\mathcal{U}(A^{**})\tilde{} \to \mathcal{F}_{n}(A_1^*)$$
$$u \mapsto \{u\}_\prime$$

   is an order isomorphism.

2. *The map*

$$\mathcal{U}(A^{**})\tilde{} \to \mathcal{F}_{w^*}(A_1^{**})$$
$$u \mapsto (\{u\}_\prime)'$$

   is an anti-order isomorphism.

1. $\mathcal{E}_n(A_1^*) = \mathcal{F}_n(A_1^*) \quad \longrightarrow \quad \mathcal{E}_n(A_1^*) = \mathcal{S}_n(A_1^*) = \mathcal{F}_n(A_1^*)$
2. $\mathcal{F}_{w^*}(A_1^{**}) = \mathcal{S}_{w^*}(A_1^{**})$

- $u \in \mathcal{U}(A^{**})$ compact-G_δ relative to A if

$$\exists a \in A, \|a\| = 1 \qquad u = u(a),$$

where $\qquad u(a) = w^* - \lim a^{2n+1}$

- $u \in \mathcal{U}(A^{**})\tilde{}$ compact relative to A if $u = 0$
or
there exists a decreasing net $(u_j)_{j\in\Lambda}$ in $\mathcal{U}(A^{**})$ of compact-G_δ tripotents relative to A such that

$$u = \wedge\{u_j : j \in \Lambda\}$$

- $\mathcal{U}_c(A)$ compact tripotents relative to A

$$\mathcal{U}_c(A)\tilde{} = \mathcal{U}_c(A) \cup \{\omega\}$$

(Edwards–Rüttimann '96)

Theorem

1. *$\mathcal{U}_c(A)\tilde{}$ together with inherited partial ordering is complete atomic lattice*
2. *The map*
$$\mathcal{U}_c(A^{**})\tilde{} \to \mathcal{S}_{w^*}(A_1^*)$$
$$u \mapsto \{u\}_\prime$$
is an order isomorphism.
3. *The map*
$$\mathcal{U}_c(A^{**})\tilde{} \to \mathcal{S}_n(A_1)$$
$$u \mapsto (\{u\}_\prime)_\prime$$
is an anti-order isomorphism.

$$(\{u\}_\prime)_\prime = (u + A_0^{**}(u)_1) \cap A \quad u \neq \omega$$

(Edwards – Fernández-Polo – Hoskin – Peralta '10)

Theorem

$$\mathcal{S}_n(A_1) = \mathcal{F}_n(A_1)$$

A JB*-triple

A^* dual of A

A_1 unit ball of A

A_1^* unit ball of A^*

F norm-closed face of A_1

$\uparrow$ norm-exposed?

$\updownarrow$ norm-semi-exposed

G weak*-closed face of A_1^*

$\uparrow$ weak*-exposed?

$\uparrow$ weak*-semi-exposed?

(Fernández-Polo – Peralta '10)

Theorem

$$\mathcal{S}_{w^*}(A_1^*) = \mathcal{F}_{w^*}(A_1^*)$$

A JB*-triple

A^* dual of A

A_1 unit ball of A

A_1^* unit ball of A^*

F norm-closed face of A_1

G weak*-closed face of A_1^*

$\uparrow$ norm-exposed?

$\Updownarrow$ norm-semi-exposed

$\uparrow$ weak*-exposed?

$\Updownarrow$ weak*-semi-exposed

A JB*-triple

A_1 unit ball of A A_1^* unit ball of A^*

F norm-closed face of A_1 G weak*-closed face of A_1^*

$\uparrow$ $\updownarrow$ $\downarrow$ $\updownarrow$

norm-exposed? norm-semi-exposed weak*-exposed weak*-semi-exposed

$\downarrow$

$a \in A,\ \|a\| = 1$ $\{a\}' = \{u(a)\},$

$\downarrow$

G weak*-exposed if and only if there exists u compact-G_δ relative to A such that

$$G = \{u\},$$

Norm-exposed faces of A_1

A JB*-triple, $\quad x \in A^*, \quad \|x\| = 1$

- $e(x)$ support tripotent of x the unique tripotent such that

$$\underbrace{(\{x\}')_\prime}_{\uparrow} = \{e(x)\}_\prime$$

  the smallest norm-closed face of A_1^* containing x

- $e_c(x)$ compact support tripotent of x the unique tripotent compact relative to A such that

$$\underbrace{(\{x\}_\prime)'}_{\uparrow} = \{e_c(x)\}_\prime$$

  the smallest weak*-closed face of A_1^* containing x

Norm-exposed faces of A_1

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- $e(x)$ support tripotent of x the unique tripotent such that

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- $e_c(x)$ compact support tripotent of x the unique tripotent compact relative to A such that

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  the smallest weak*-closed face of A_1^* containing x

Norm-exposed faces of A_1

1/3

Theorem

Let u be a tripotent in A^{**} compact relative to A. Then, the following are equivalent.

1. *The norm-semi-exposed face $(\{u\}_{\prime})_{\prime}$ of the unit ball A_1 in A is norm-exposed.*
2. *There exists an element x of norm one in A^* such that u coincides with the compact support tripotent $e_c(x)$ of x.*

Norm-exposed faces of A_1

B JBW*-triple $u, v \in \mathcal{U}(B)$ tripotents

- Peirce decomposition

$$B = B_0(u) \oplus B_1(u) \oplus B_2(u)$$

$$B_j(u) = \{a \in B\colon D(u,u)a = \tfrac{1}{2}ja\} \qquad j = 0,1,2 \qquad D(u,u)a = \{u\,u\,a\}$$

- u and v are said to be orthogonal if v lies in $B_0(u)$ $\quad u \perp v$

Norm-exposed faces of A_1

- A pair x and y of elements A_1^*, each of norm one, is said to be orthogonal if $e(x) \perp e(y)$

2/3

Theorem

Let u be a tripotent in A^{**} compact relative to A. Then, the following are equivalent.

1. *The norm-semi-exposed face $(\{u\}_\prime)_\prime$ of the unit ball A_1 in A is norm-exposed.*
2. *There exists an element x of norm one in A^* such that u coincides with the compact support tripotent $e_c(x)$ of x.*
3. *There exists a countable maximal family $\{x_j : j \in \Lambda\}$ of orthogonal elements of A^* of norm one such that*

$$u = \vee_{j\in\Lambda}^{c} e_c(x_j).$$

Norm-exposed faces of A_1

3/3

Theorem

Let u be a tripotent in A^{**} compact relative to A. Then, the following are equivalent.

1. *The norm-semi-exposed face $(\{u\}_{\prime})_{\prime}$ of the unit ball A_1 in A is norm-exposed.*
2. *There exists an element x of norm one in A^* such that u coincides with the compact support tripotent $e_c(x)$ of x.*
3. *There exists a countable maximal family $\{x_j : j \in \Lambda\}$ of orthogonal elements of A^* of norm one such that*

$$u = \vee^c_{j\in\Lambda} e_c(x_j).$$

4. *The subset $\{u\}$ of the closed unit ball $(P_2(u)A)_1$ in the complex Banach space $P_2(u)A$ is a norm-exposed face of $(P_2(u)A)_1$.*

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