Integrable Clusters

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- A. Berenstein, J, Greenstein, D. Kazhdan, *Comptes rendus Mathematique* vol. 353, **5** (2015).

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Integrable systems

Informally, a *(completely) integrable system* in a given a Poisson algebra A is any maximal Poisson-commutative subalgebra A_0.

In particular, a Hamiltonian H is any element of A_0 such that A_0 is the Poisson centralizer of H.

It is well-known that if the bracket on A is symplectic, then $\dim\ A_0 = \frac{1}{2}\dim\ A$ and the map $Spec A \twoheadrightarrow Spec A_0$ is a Lagrangian foliation.

Problem. Classify all integrable systems in A.

Cluster structures

Upper bounds. Given a field F, *char* $F = 0$, a *cluster* $\mathbf{x} = (x_1, \ldots, x_m)$ is any algebraically independent set in F. Each cluster $\mathbf{x}$ defines a Laurent polynomial algebra $L_{\mathbf{x}} = \mathbb{Q}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] = \oplus_{a \in \mathbb{Z}^m} \mathbb{Q}x^a$.

Given $n \leq m$, a *seed* is a pair $(\mathbf{x}, \tilde{B})$, where $\tilde{B} = (b_1 \cdots b_n)$ is an integer $m \times n$ matrix ($\tilde{B}$ is called an *exchange matrix*).

Define the *upper bound* algebra $U(\mathbf{x}, \tilde{B}) \subset L_{\mathbf{x}}$ by

$$U(\mathbf{x}, \tilde{B}) := \bigcap_{k=1}^{n} U_k(\mathbf{x}, \tilde{B}),$$

where $U_k(\mathbf{x}, \tilde{B})$ is the subalgebra of L generated by $\mathbf{x}$, all x_i^{-1}, $i \neq k$ and $x_k' = x_k^{-1}(x^{[b_k]_+} + x^{[-b_k]_+})$. Here we abbreviate $[(a_1, \ldots, a_m)]_+ = (\max(0, a_1), \ldots, \max(0, a_m)]$.

Mutations. For each seed $(\mathbf{x}, \tilde{B})$ and $k = 1, \ldots, n$ define $\mu_k(\mathbf{x}, \tilde{B}) := (\mathbf{x} \setminus \{x_k\} \cup \{x_k'\}, \mu_k(\tilde{B}))$, where

$$\mu_k(\tilde{B})_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + \frac{|b_{ik}| b_{kj} + b_{ik} |b_{kj}|}{2} & \text{otherwise} \end{cases}$$

Theorem 1 (BFZ 2005) *Suppose that rank* $\tilde{B} = n$ *and* $\tilde{B}|_{[1,n]\times[1,n]}$ *is skew-symmetrizable. Then* $\mu_k(\tilde{B})$ *satisfies same properties and*

$$U(\mu_k(\mathbf{x}, \tilde{B})) = U(\mathbf{x}, \tilde{B})$$

for $k = 1, \ldots, n$.

In particular, for any sequence $(\mathbf{y}, \tilde{B}') = \mu_{i_\ell} \cdots \mu_{i_1}(\mathbf{x}, \tilde{B})$, *each* y_k *belongs to the Laurent polynomial algebra* $L_{\mathbf{x}}$.

Poisson clusters. Given an $m \times m$ skew symmetric matrix Λ, equip $L_{\mathbf{x}}$ with the (log-canonical) Poisson bracket via $\{x_i, x_j\} = \lambda_{ij} x_i x_j$.

Theorem 2 Suppose that $\tilde{B}$ has no zero columns. Then $U(\mathbf{x}, \tilde{B})$ *is a Poisson subalgebra of* $L_{\mathbf{x}}$ *iff* $\tilde{B}^T \Lambda = (D\ \mathbf{0})$ *for some* $D = diag(d_1, \ldots, d_n)$.

We say that Λ as in Theorem 2 is *compatible* with $\tilde{B}$ if $d_1 \neq 0, \ldots, d_n \neq 0$.

Lemma If $\tilde{B}$ *admits a compatible* Λ *then* $\tilde{B}$ *is as in Theorem 1.*

Main example If $B, C, D \in Mat_{n\times n}(\mathbb{Z})$, $D = diag(d_1, \ldots, d_n)$, $(DB)^T = -DB$, det $C \neq 0$, det $D \neq 0$, then the $2n \times 2n$ matrix
$$\Lambda = \Lambda_{DB,C} = \begin{pmatrix} \mathbf{0} & -DC^{-1} \\ (C^{-1})^T D & -(C^{-1})^T DBC^{-1} \end{pmatrix}$$
is compatible with the exchange matrix $\tilde{B} = \begin{pmatrix} B \\ C \end{pmatrix}$.

Integrable seeds

We say that $(\mathbf{X}, \tilde{B})$ is an *integrable seed* if $\tilde{B} = \begin{pmatrix} B \\ C \end{pmatrix}$ as in Main example.

Lemma *For each integrable seed $(\mathbf{x}, \tilde{B})$ the algebra $A_0 = \mathbb{Q}[x_1, \ldots, x_n]$ is an integrable system in the Poisson algebra $A = U(\mathbf{x}, \tilde{B}) \subset L_{\mathbf{x}}$ whose Poisson bracket given by $\Lambda_{DB,C}$.*

Problem. Find all integrable seeds $(\mathbf{x}', \tilde{B}')$ mutation equivalent to a given integrable seed $(\mathbf{x}, \tilde{B})$.

Main Theorem *Let $(\mathbf{x}, \tilde{B})$ is be a principal (i.e., $C = I_n$, the identity matrix) integrable seed. Then all seeds $(\mathbf{x}', \tilde{B}')$ mutation equivalent to $(\mathbf{x}, \tilde{B})$ are integrable.*

Proof uses the *sign coherence conjecture* (now theorem), which asserts that in each exchange matrix $\begin{pmatrix} B' \\ C \end{pmatrix} = \mu_{i_\ell} \cdots \mu_{i_1} \begin{pmatrix} B \\ I_n \end{pmatrix}$ each column of C is either in $(\mathbb{Z}_{\geq 0})^n$ or in $-(\mathbb{Z}_{\geq 0})^n$.

Quantum story

A quantum *integrable system* in a given algebra A is any maximal commutative subalgebra A_0.

Given a skew field $\mathcal{F}$ and a central transcendental element $q^{\frac{1}{2}}$ in $\mathcal{F}$, a quantum cluster $\mathbf{X} = (X_1, \ldots, X_m)$ is any algebraically independent subset such that

$$X_i X_j = q^{\lambda_{ij}} X_j X_i$$

for all $1 \leq i, j \leq m$ and some skew-symmetric matrix $\Lambda_{\mathbf{X}} = (\lambda_{ij})$. This defines a quantum torus $L_{\mathbf{X}} = \bigoplus_{a \in \mathbb{Z}^m} \mathbb{Q}(q^{\frac{1}{2}}) X^a$, where we abbreviate $X^a = q^{\frac{1}{2}\sum_{i<j} \lambda_{ji} a_i a_j} X_1^{a_1} \cdots X_m^{a_m}$ for $a \in \mathbb{Z}^m$ (so that $X^a X^b = q^{\frac{1}{2}\Lambda(a,b)} X^{a+b}$).

A pair $(\mathbf{X}, \tilde{B})$ is a *quantum seed* if $\Lambda_{\mathbf{X}}$ is compatible with $\tilde{B}$. The *quantum upper cluster algebra* $U(\mathbf{X}, B)$ is the intersection $\bigcap_{k=1}^{n} U_k(\mathbf{X}, \tilde{B})$, where $U_k(\mathbf{X}, \tilde{B})$ is the subalgebra of the quantum torus $L_{\mathbf{X}}$ generated by $\mathbf{X}$, all X_i^{-1}, $i \neq k$ and $X_k' = X^{[b_k]_+ - e_k} + X^{[-b_k]_+ - e_k}$. Here we abbreviate $X^{e_k} = X_k$.

For each quantum seed $(\mathbf{X}, \tilde{B})$ and $k = 1, \dots, n$ define $\mu_k(\mathbf{X}, \tilde{B}) := (\mathbf{X} \setminus \{X_k\} \cup \{X_k'\}, \mu_k(\tilde{B}))$.

Theorem 3 (BZ 2005) *For any quantum seed* $(\mathbf{X}, B)$ *and* $k = 1, \dots, n$ *one has:*

(a) $\mu_k(\mathbf{X}, B)$ *is also a quantum seed.*

(b) $U(\mu_k(\mathbf{x}, \tilde{B})) = U(\mathbf{x}, \tilde{B})$.

We say that $(\mathbf{X}, \tilde{B})$ is an *quantum integrable seed* if $\tilde{B} = \begin{pmatrix} B \\ C \end{pmatrix}$ as in Main example.

Lemma *For each quantum integrable seed* $(\mathbf{X}, \tilde{B})$ *the algebra* $A_0 = \mathbb{Q}(q)[X_1, \ldots, X_n]$ *is an quantum integrable system in* $A = U(\mathbf{X}, \tilde{B})$*.*

q-Main Theorem *Let* $(\mathbf{X}, \tilde{B})$ *be an integrable principal quantum seed. Then all quantum seeds mutation equivalent to* $(\mathbf{X}, \tilde{B})$ *are also integrable.*