Opers, spectral networks and the T3 theory

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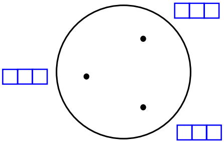
based on arXiv:1312.2979 with A. Neitzke
and work in progress

$\mathcal{N} = 2$ theory

punctured Riemann surface C	$\longleftrightarrow$	four-dimensional quantum field theory $T[C]$
$\{\varphi_2, \ldots, \varphi_K\}$	$\longleftrightarrow$	vacua

$\varphi_k = u_k(z)(dz)^k$
holomorphic k-differential

$T[3]$ theory

punctured Riemann surface C	$\longleftrightarrow$	four-dimensional quantum field theory $T[C]$
K	$\longleftrightarrow$	rank
$K > 2$	$\longleftrightarrow$	strongly coupled non-Lagrangian
	$\longleftrightarrow$	$T[3]$ theory

Proposal [Nekrasov-Rosly-Shatashvili]

generating function of space of $PSL(2)$ opers on C	$\longleftrightarrow$	superpotential of two-dimensional field theory (emerging from $T[C]$ on $\mathbb{R}^2_\epsilon \times \mathbb{R}^2$)

$PSL(2)$ Oper

- $PSL(2)$ oper is locally a linear differential operator

$$D = d^2 + a,$$

that acts on $(-\frac{1}{2})$-differentials on C.

- Coefficient a transforms as a Schwartzian derivative.
- If we write

$$a = \{f, z\}(dz)^2,$$

then f defines a projective structure on C.

Example: hypergeometric oper

- C is the three-punctured sphere.
- The canonical $PSL(2)$ oper on C is

$$D = d^2 + t_0(z),$$

with the uniformization term

$$t_0(z) = \frac{1 - z + z^2}{2z^2(z-1)^2} dz^2.$$

- A generic $PSL(2)$ oper on C is of the form

$$D = d^2 + t_0(z) + \varphi_2(z),$$

where $\varphi_2(z)$ is a holomorphic quadratic differential on C (with regular poles at the punctures).

Hitchin moduli space

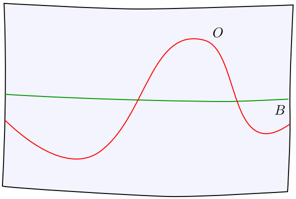
- Consider the moduli space $\mathcal{M}_\zeta[C]$ of solutions to the Hitchin equations on C:

$$\begin{aligned} F_D - [\Phi, \Phi] &= 0 \\ D\Phi &= 0 \\ D * \Phi &= 0 \end{aligned}$$

- For $\zeta = 0$ this is the Hitchin complex integrable system.
- For $\zeta \neq 0$ it is the moduli space $\mathcal{M}_{flat}$ of flat $G_\mathbb{C}$ connections on C.

Space of opers

- The space of opers is a Lagrangian of $\mathcal{M}_{flat}$



- Given choice of Darboux coordinates $\{\alpha, \beta\}$, we may consider its generating function

$$\frac{\partial W_O(\alpha)}{\partial \alpha} = \beta. \tag{1}$$

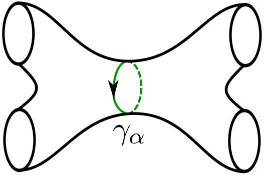
Proposal [Nekrasov-Rosly-Shatashvili]

generating function $W_O(\alpha)$ of space of $PSL(2)$ opers on C	$\longleftrightarrow$	superpotential of two-dimensional field theory
		(emerging from $T[C]$ on $\mathbb{R}^2_\epsilon \times \mathbb{R}^2$)

if $\{\alpha, \beta\}$ are complex Fenchel-Nielsen coordinates on C.

Complex Fenchel-Nielsen coordinates

- Coordinates on the $PSL(2, \mathbb{C})$ character variety.
- Defined with respect to pair of pants decomposition of C.



- To each pants curve γ_α we associate two coordinates.
- The length coordinate α is defined as

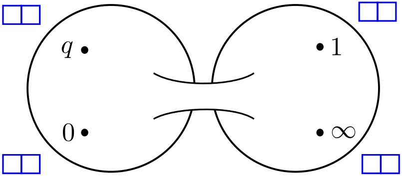
$$\operatorname{Tr} M_{\gamma_\alpha} = e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}.$$

- The twist coordinate β can be defined through a Hamiltonian flow.

Example

four-punctured sphere $\longleftrightarrow$ $SU(2)$ gauge theory with four hypers

complex structure q $\longleftrightarrow$ coupling constant $q = \exp(\tau)$



Example

- When $q \to 0$ the $PSL(2)$ oper can be approximated by a hypergeometric oper.
- We find that $W_O(\alpha)$ is indeed the twisted superpotential for $T[C]$.

Higher rank generalization

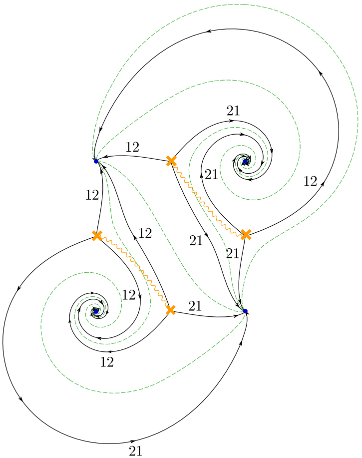
- [Work in progress with Andrew Neitzke]
- Find higher rank generalization of complex Fenchel-Nielsen coordinates.

Spectral Network

- Fix C and set of k-differentials $\{\varphi_2, \dots, \varphi_K\}$.
- Write
$$0 = w^K - \sum_{k=2}^{K} \varphi_k w^{K-k} = \prod_{i=1}^{K} (w - \lambda_i)$$
- Spectral network is a collection of trajectories on C, such that
$$\lambda_i(v) - \lambda_j(v) \in \mathbb{R}$$
for any non-zero tangent vector v.
- When $K = 2$ this is just the critical graph of a foliation defined by the differential φ_2.

FG network for $K = 2$

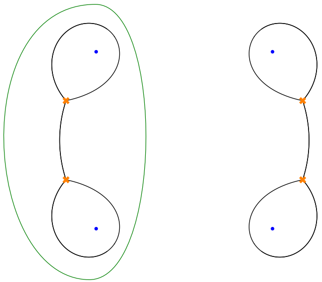
- Generically all leaves end on punctures:



- Such a network is dual to an ideal triangulation of C.

FN network for $K = 2$

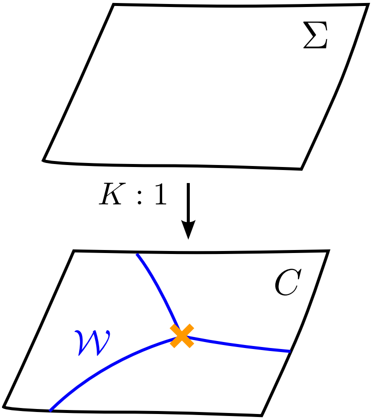
- Other extreme, no leaves ending on punctures and all leaves compact:



- Such a network is dual to a pants decomposition of C and generated by a Strebel differential:

$$\oint_{\gamma_\alpha} \sqrt{\varphi_2} \in \mathbb{R}.$$

Nonabelianization

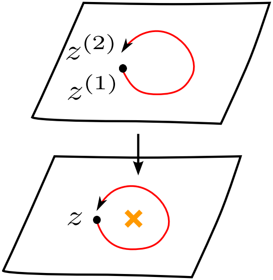


Nonabelianization
is a map $\psi_{\mathcal{W}}$:

$$\mathcal{M}_{flat}(\Sigma, \mathbb{C}^*)$$
$$\downarrow$$
$$\mathcal{M}_{flat}(C, G_{\mathbb{C}}).$$

Idea

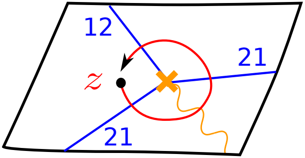
- Given flat $GL(1)$ connection ∇^{ab} in a complex line bundle over Σ, can we push-forward to find a flat $GL(K)$ connection ∇ on C?
- Not quite. We can find $\nabla = \pi_* \nabla^{ab}$ on a complement of the branch locus, but it does not extend smoothly. Reason:



When we go around branch point the sheets get permuted, this shows up as nontrivial monodromy.

Details

- Nonabelianization is a recipe for fixing up the monodromy using the data of a spectral network $\mathcal{W}$.
- We cut the surface C into pieces $C \backslash \mathcal{W}$.
- On each of the pieces we define ∇ as $\pi_* \nabla^{ab}$.
- We reglue along the gluing lines with non-trivial automorphisms of $\pi_* \nabla^{ab}$.
- These are given by unipotent matrices $1 + e$ with $e : \mathcal{L}_i \to \mathcal{L}_j$ when the label of the wall is ij.



Spectral coordinates

- For many $\mathcal{W}$ (including the examples we discussed) the map $\psi_{\mathcal{W}}$ is invertible and there is also a canonical way to obtain a connection ∇^{ab} on Σ from a non-abelian connection ∇ on C.
- Given the $GL(1)$ connection ∇^{ab} we can construct interesting numbers

$$\chi_\gamma = \mathrm{Hol}_\gamma \nabla^{ab} \in \mathbb{C}^*$$

  for any $\gamma \in H^1(\Sigma, \mathbb{Z})$.
- These numbers are thus coordinates on the moduli space of non-abelian flat connections.
- We call them spectral coordinates.

Darboux coordinates $K = 2$

- If $\mathcal{W}$ is generated by a generic quadratic differential, then we recover the Fock-Goncharov coordinates.
- If $\mathcal{W}$ is generated by a Strebel differential, then we recover the complex Fenchel-Nielsen coordinates.

Higher rank complex Fenchel-Nielsen coordinates

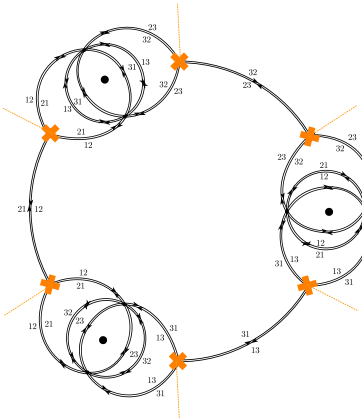
- A generalized Strebel differential is a tuple of differentials such that
$$\oint_{\gamma_A} \lambda \in \mathbb{R}$$
for a choice of A-cycles $\{\gamma_A\}$ on Σ.

- Define a rank K Fenchel-Nielsen network as a spectral network generated by a generalized Strebel differential.

- We propose that rank K complex Fenchel-Nielsen coordinates are the spectral coordinates for a rank K Fenchel-Nielsen network.

Example $T[3]$

- Example of rank 3 Fenchel-Nielsen network:



- To do: find superpotential of T[3] theory.
- Puzzle: many inequivalent networks!