On some approaches of topological complexity

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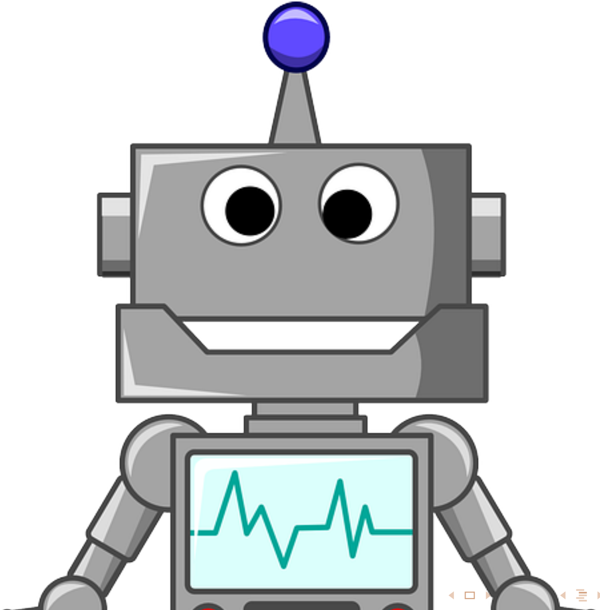
Contents of the talk

First part: Topological complexity of robot motion planning



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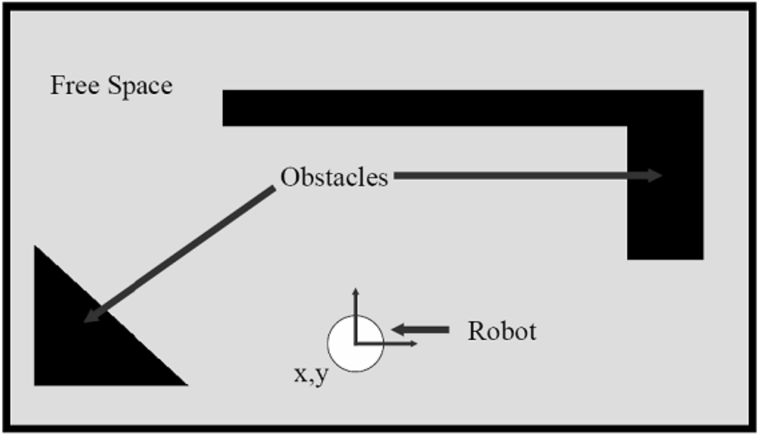
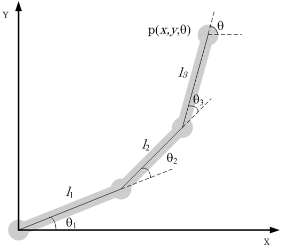
- **States** (or **configurations**) of the system S correspond to **points** $A \in X$
- **Motions** of the system from the state A to the state B correspond to **paths** in X joining A to B.

Some examples of configuration spaces:

- A robot moving on a room with obstacles



• A *robot arm* consisting of several bars connected by revolving joins. We allow self-intersections of the arm



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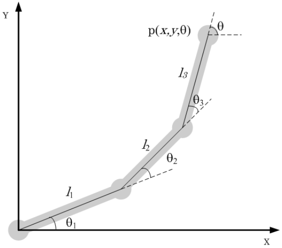

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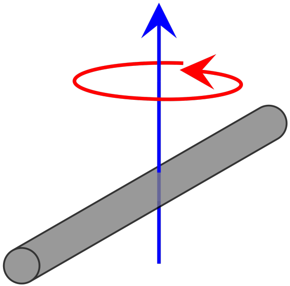
In the spacial case we have: $X = S^2 \times S^2 \times ... \times S^2$

- A *flying robot* that can translate and rotate.

A configuration requires 6 parameters: (x, y, z) for translation, and Euler angles (α, β, γ) for rotation.
The configuration space is $X = \mathbb{R}^3 \times SO(3)$.

- A rigid bar moving freely in the 3-space, where the center is fixed



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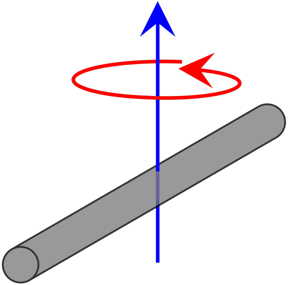

The configuration space is $\mathbb{RP}^2$ (the real projective plane)

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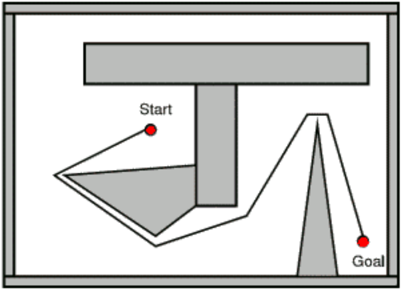

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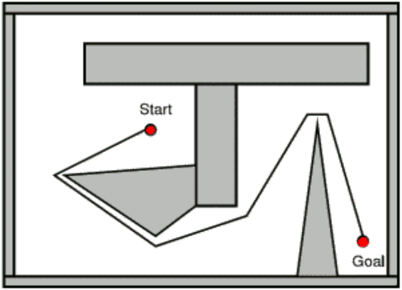

Motion planning has several robotics applications: automation (or automatic control), robotic surgery, architectural design, video game artificial intelligence, the study of biological molecules, et cetera...

Motion planning algorithm

In terms of the configuration space X the motion planning algorithm:

- **Input**: a point $(A, B) \in X \times X$
- **Output**: a path $\alpha : [0, 1] \to X$ such that $\alpha(0) = A$ and $\alpha(1) = B$.

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In these terms, a motion planning algorithm is precisely a **section** (not necessarily continuous) of π. That is, a map

$$s : X \times X \to X^I$$

such that $\pi \circ s = id$

Continuity of a motion planning algorithm s is desired. It means that the suggested route $s(A,B)$ of going from A to B depends continuously on the states A and B

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Consequence

In general, motion planning algorithms have discontinuities.

We can consider **local** continuous sections of π. These are maps defined on an open subset $U \subset X \times X$

$$s : U \to X^I$$

such that $\pi \circ s = inc : U \hookrightarrow X \times X$.

Topological complexity of a topological space

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Definition (Farber)

The (normalized) **topological complexity** of a topological space X, $\mathrm{TC}(X)$, is the least non-negative integer k such that $X \times X$ can be covered by $k+1$ open subsets

$$X \times X = U_0 \cup U_1 \cup \ldots \cup U_k$$

on each of which $\pi : X^I \to X \times X$ admits a *local continuous section*.

A general invariant

Definition

The **sectional category** (or Schwarz genus) of a fibration $p : E \twoheadrightarrow B$, $\mathrm{secat}(p)$, is the least integer k such that B can be covered by $k+1$ open subsets $B = U_0 \cup U_1 \cup \ldots \cup U_k$ on each of which there exists a local continuous section $s_i : U_i \to E$

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Equivalently, we can take the associated fibration of f:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\simeq"'] & & Y \\ & Z \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

and define $\mathrm{secat}(f) := \mathrm{secat}(p)$.

Topological complexity is a sectional category

$\mathrm{TC}(X) = \mathrm{secat}(\pi : X^I \to X \times X)$. Observe that we can also consider

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Useful property for computations

Let $p : E \to B$ be any map and consider $p^* : H^*(B) \to H^*(E)$ the homomorphism induced in cohomology of p. If $x_1, ..., x_k \in H^*(B)$ are such that $p^*(x_i) = 0$ and $x_1 \cup ... \cup x_k \neq 0$, then $\mathrm{secat}(p) \geq k$.

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The property for topological complexity

Taking cohomology with coefficients in any field we have that

$$(\Delta_X)^* = \cup : H^*(X) \otimes H^*(X) \to H^*(X)$$

is precisely the cup product. The kernel of $\cup$ is called the *ideal of zero-divisors* of $H^*(X)$. Therefore $\mathrm{TC}(X) \geq \mathrm{nil}\,\ker(\cup)$

Back to topological complexity

Basic properties:

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Theorem (Farber-Tabachnikov-Yuzvinski (2003))

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Strategy

One way of dealing with topological complexity is to consider approximations that, in some sense, are more manageable and therefore more computable.

An auxiliary invariant: Lusternik-Schnirelmann category

Definition

The L.S. **category** of a space X, $\mathrm{cat}(X)$, is the least non-negative integer k such that X admits an open cover $X = U_0 \cup U_1 \cup \ldots \cup U_k$ where each U_i is contractible in X

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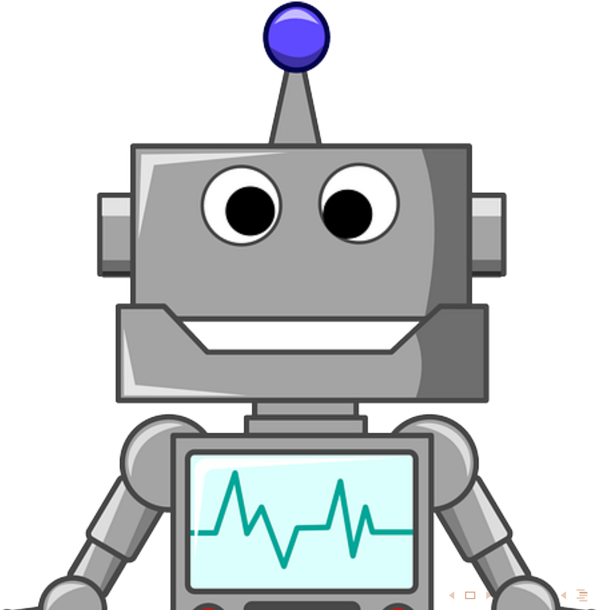
- $\mathrm{cat}(X) \geq \mathrm{cuplenght}(X)$
- Let X be any path-connected space. Then

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq 2\,\mathrm{cat}(X)$$

Second part: Approaches of TC



First approach: Weak topological complexity

Let $p : E \hookrightarrow B$ be a cofibration. We consider:

$$T^n(p) := \{(b_0, b_1, ..., b_n) \in B^{n+1} : b_i \in E \text{ for some } i\}$$

and denote $k_n : T^n(p) \hookrightarrow B^{n+1}$ the natural inclusion.

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Theorem (Whitehead characterization of sectional category)

Let p be any map, where B is a normal space. Then $\mathrm{secat}(p) \leq n$ if and only if there is, up to homotopy, a lift

$$\begin{tikzcd} & T^n(p) \arrow[d, "k_n"] \\ B \arrow[ur, dotted] \arrow[r, "\Delta_{n+1}"'] & B^{n+1} \end{tikzcd}$$

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$$T^n(\Delta_X) = \{(y_0, y_1, \ldots, y_n) \in (X \times X)^{n+1} : y_i \in \Delta(X) \text{ for some } i\}$$

Moreover, $\mathrm{TC}(X) \leq n$ if and only if

$$\begin{tikzcd} & T^n(\Delta_X) \arrow[d, hook] \\ X \times X \arrow[ur, dotted] \arrow[r, "\Delta_{n+1}"'] & (X \times X)^{n+1} \end{tikzcd}$$

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Observation

If X is a CW-complex or a topological manifold, then the diagonal map Δ_X is a cofibration. The spaces for which this fact holds are called *locally equiconnected*.

Weak sectional category

Definition

Let $p : E \to B$ be any map. The **weak sectional category** of p, $\mathrm{wsecat}(p)$ is the least integer n such that the composition $l_n \Delta_{n+1} \simeq *$ is null-homotopic:

$$\begin{tikzcd} & B \arrow[d, "\Delta_{n+1}"'] \arrow[dr, dotted, "*"] & \\ T^n(p) \arrow[r, "k_n"'] & B^{n+1} \arrow[r, "l_n"'] & C_{k_n} \end{tikzcd}$$

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where the bottom row is the homotopy cofibre of k_n.

- $\mathrm{wsecat}(p) \leq \mathrm{secat}(p)$ (use Whitehead characterization)
- When $p = * : * \to X$, then we recover the usual notion of *weak category*, $\mathrm{wcat}(X)$, in the sense of Berstein-Hilton.

Properties of weak sectional category

Theorem (L.Vandembroucq, G.-C.)

Let $p : E \to B$ be any map, and C_p denote its homotopy cofibre. Then the following hold:

- $\mathrm{wsecat}(p) \leq \mathrm{wcat}(B)$
- $\mathrm{wsecat}(p) \geq \mathrm{nil}\ker p^*$
- $\mathrm{wcat}(C_p) - 1 \leq \mathrm{wsecat}(p) \leq \mathrm{wcat}(C_p)$
- If the map p admits a *homotopy retraction*, then

$$\mathrm{wsecat}(p) = \mathrm{wcat}(C_p)$$

  Moreover $\mathrm{nil}\ker p^* = \mathrm{cuplength}(C_p)$.

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$$\mathrm{wsecat}(p) = \mathrm{wcat}(C_p)$$

Moreover $\mathrm{nil}\ker p^* = \mathrm{cuplength}(C_p)$.

Weak topological complexity

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Corollary (Properties of weak topological complexity)

Let X be any space. Then, if C_{Δ_X} denotes the homotopy cofibre of the diagonal map $\Delta_X : X \to X \times X$ we have:

- $\mathrm{wTC}(X) \geq \mathrm{nil}\ker(\cup) = \mathrm{cuplength}(C_{\Delta_X})$
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Comment

The inequality $\mathrm{wTC}(X) \geq \mathrm{nil}\ker(\cup)$ can be strict.
For instance, if $X = S^3 \cup_\alpha e^7$ is the 7-skeleton of $\mathrm{Sp}(2)$, then it can be proved that $\mathrm{nil}\ker(\cup) = 2$ and $\mathrm{wTC}(X) = 3$.

Second approach: the category of C_{Δ_X}

We have seen that $\mathrm{wTC}(X) = \mathrm{wcat}(C_{\Delta_X})$. Is it true that $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$?

Second approach: the category of C_{Δ_X}

We have seen that $\mathrm{wTC}(X) = \mathrm{wcat}(C_{\Delta_X})$. Is it true that $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$?

Arkowitz and Strom proved that if $A \xrightarrow{h} Y \xrightarrow{\alpha} C_h$ is a homotopy cofibre, then $\mathrm{cat}(C_h) \leq \mathrm{secat}(h) + 1$. In our case we have:

$$\mathrm{cat}(C_{\Delta_X}) \leq \mathrm{TC}(X) + 1$$

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Using a Ganea-type characterization of sectional category (iterated fibred joins of the map) and classical homotopy methods we obtain:

Theorem (L.Vandembroucq, G.C.)

Let X be a $(q-1)$-connected finite dimensional CW-complex $(q \geq 1)$.

- If $\dim(X) \leq q(\mathrm{TC}(X)+1) - 2$, then $\mathrm{cat}(C_{\Delta_X}) \leq \mathrm{TC}(X)$.
- If $2\dim(X) \leq 2q - 2 + q\mathrm{cat}(C_{\Delta_X})$, then $\mathrm{TC}(X) \leq \mathrm{cat}(C_{\Delta_X})$.

Examples in which $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$

- $\mathrm{TC}(S^n) = \mathrm{cat}(C_{\Delta_{S^n}})$
- If X is an H-space, then $\mathrm{TC}(X) = \mathrm{cat}(X) = \mathrm{cat}(C_{\Delta_X})$
- If X is a closed, 1-connected symplectic manifold. Then $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$
- If $X = \Sigma_g$ is a compact orientable surface of genus g, then $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$
- If X is any finite graph, then $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$
- If $X = F(\mathbb{R}^m, n)$ is the space of configurations of n distinct points in $\mathbb{R}^m$

$$F(\mathbb{R}^m, n) = \{(x_1, \ldots, x_n) \in (\mathbb{R}^m)^n : x_i \neq x_j \text{ for } i \neq j\}$$

with $n, m \geq 2$, then $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$

The case of real projective spaces

In this case we need more sophisticated techniques. The key point is the non-existence of elements of Hopf invariant one (Adams, F.J.).

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Let $n \geq 0$ be any integer. Then $\mathrm{TC}(\mathbb{RP}^n) = \mathrm{cat}(C_{\Delta_{\mathbb{RP}^n}})$.

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Open problem

Is it true that $\mathrm{TC}(X) = \mathrm{cat}(C_{\Delta_X})$ for any space X?

So far we haven't found any counterexamples!

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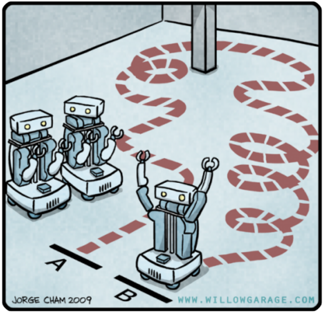

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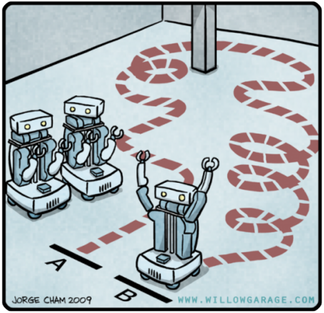

"HIS PATH-PLANNING MAY BE SUB-OPTIMAL, BUT IT'S GOT FLAIR."

We want s to be *optimal* by requiring $s(A, A)$ to be the static path at A (i.e., no movement from A to A!!!)

Definition (Iwase-Sakai)

The **monoidal topological complexity** of a space X, $\mathrm{TC}^M(X)$, is the least non-negative integer k such that $X \times X$ admits an open cover

$$X \times X = U_0 \cup U_1 \cup \ldots \cup U_k$$

such that for any i, $\Delta_X(X) \subset U_i$ and $\pi : X^I \to X \times X$ admits a local continuous section s_i satisfying $s_i(A, A) = c_A$ (i.e., the constant path at A).

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Theorem (Dranishnikov)

If X is a $(q-1)$-connected CW-complex and $\dim(X) \leq q(\mathrm{TC}(X)+1) - 2$, then $\mathrm{TC}(X) = \mathrm{TC}^M(X)$.

Let $p : E \hookrightarrow B$ be a cofibration. Recall the *n-th sectional fat wedge*:

$$T^n(p) = \{(b_0, b_1, ..., b_n) \in B^{n+1} : b_i \in E \text{ for some } i\}$$

Definition (Doeraene, El Haouari)

Let $p : E \hookrightarrow B$ be a cofibration. The **relative category** of p, $\mathrm{relcat}(p)$ is the least non-negative integer n such that there is a map $\phi : B \to T^n(p)$ such that $\phi p = \tau_n$ and $k_n \phi \simeq \Delta_{n+1}$ rel. E

$$\begin{tikzcd} E \arrow[r, "\tau_n"] \arrow[d, hook, "p"'] & T^n(p) \arrow[d, hook, "k_n"] \\ B \arrow[ur, dotted, "\phi"] \arrow[r, "\Delta_{n+1}"'] & B^{n+1} \end{tikzcd}$$

If p is any map, then $\mathrm{relcat}(p) := \mathrm{relcat}(p')$ where p' is the cofibration associated to p.

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For $p = \Delta_X$ we have Iwase-Sakai's conjecture $\mathrm{TC}(X) = \mathrm{TC}^M(X)$

Weak relative category

Definition (Weak relative category)

Let $p : E \hookrightarrow B$ be a cofibration. The **weak relative category** of p, $\mathrm{wrelcat}(p)$, is the least integer n such that the composition $l_n \Delta_{n+1} \simeq *$ rel. E:

$$\begin{tikzcd} & B \arrow[d, "\Delta_{n+1}"'] \arrow[dr, dotted, "*"] & \\ T^n(p) \arrow[r, "k_n"'] & B^{n+1} \arrow[r, "l_n"'] & C_{k_n} \end{tikzcd}$$

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Let $p : E \hookrightarrow B$ be a cofibration and $C_p = B/E$ its homotopy cofibre. Then the following chain of inequalities holds:

$$\mathrm{cuplength}(C_p) \leq \mathrm{wcat}(C_p) = \mathrm{wrelcat}(p) \leq \mathrm{relcat}(p)$$

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Corollary

If X is a CW-complex, then $\mathrm{wTC}(X) = \mathrm{wTC}^M(X)$. Moreover, this number agrees with $\mathrm{wcat}(C_{\Delta_X})$.

Thanks for listening!