The colored $\mathfrak{sl}_3$ homology

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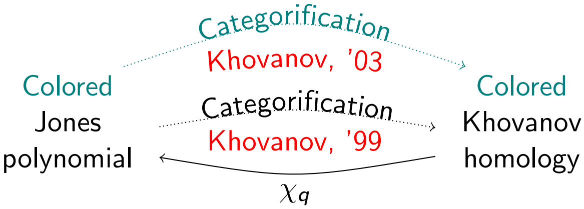
DER FORSCHUNG | DER LEHRE | DER BILDUNG

AMS-EMS-SPM Joint Meeting, Porto, 11 June 2015

Jones polynomial $\xrightarrow{\text{Categorification}}$ Khovanov homology

Khovanov, '99

Khovanov homology $\xrightarrow{\chi_q}$ Jones polynomial


Categorification
Khovanov, '03
Colored
Jones
polynomial
Categorification
Khovanov, '99
χ_q
Colored
Khovanov
homology

Categorification

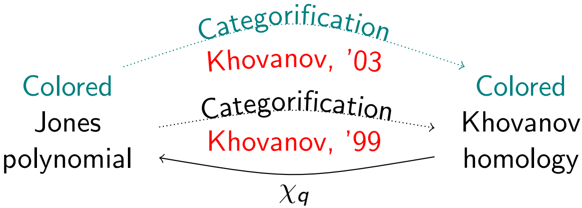
Khovanov, '03

Colored Jones polynomial —Categorification (Khovanov, '99)→ Colored Khovanov homology

Colored Khovanov homology —χ_q→ Colored Jones polynomial

$\mathfrak{sl}_3$ polynomial —Categorification (Khovanov, '03)→ $\mathfrak{sl}_3$ homology

$\mathfrak{sl}_3$ homology —χ_q→ $\mathfrak{sl}_3$ polynomial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Categorification
This talk
Colored
$\mathfrak{sl}_3$
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Khovanov's strategy for the colored Jones polynomial

- Build a nice resolution of simple $U_q(\mathfrak{sl}_2)$-modules,

$$J_n(K) = \sum_{k \in \mathbb{N}} (-1)^k \binom{n-k}{k} J(K_{//n-2k});$$

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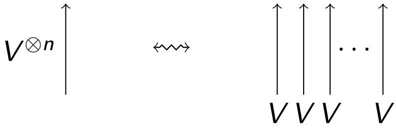
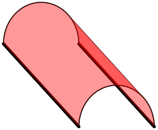
- Plug this resolution into Khovanov homology via cablings and cobordisms:

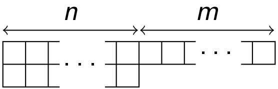




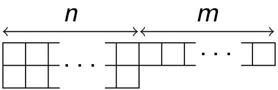
$U_q(\mathfrak{sl}_3)$ is the unital $\mathbb{C}(q^{\frac{1}{6}})$-algebra presented by:

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \qquad K_iK_j = K_jK_i,$$
$$K_iE_i = q^2E_iK_i, \qquad K_iF_i = q^{-2}F_iK_i,$$
$$K_iE_j = qE_jK_i, \qquad K_iF_j = q^{-1}F_jK_i,$$
$$E_iF_i - F_iE_i = \frac{K_i - K_i^{-1}}{q - q^{-1}}, \qquad E_iF_j = F_jE_i,$$
$$E_i{}^2E_j - [2]E_iE_jE_i + E_jE_i{}^2 = 0, \quad F_i^2F_j - [2]F_iF_jF_i + F_jF_i{}^2 = 0,$$
$$i, j \in \{1, 2\}, \quad i \neq j.$$

$$V_{m,n} \leftrightsquigarrow$$



$$V_+ = V_{1,0},\ V_- = V_{0,1} = V_+^*.$$

$V_{m,n} \leftrightsquigarrow$ 

, $\quad V_+ = V_{1,0}, \, V_- = V_{0,1} = V_+^*.$

Pieri's Rule

$$V_+ \otimes V_{m,n} \simeq V_{m+1,n} \oplus V_{m-1,n+1} \oplus V_{m,n-1},$$
$$V_- \otimes V_{m,n} \simeq V_{m,n+1} \oplus V_{m+1,n-1} \oplus V_{m-1,n}.$$

$$\langle K; V_{m,n}\rangle = \sum_{\substack{(i,j,k,l)\in\mathbb{N}^4\\ \delta\in\{0,1\}}} (-1)^{\delta+i+k} \binom{m-\delta-i-2j}{i \quad j \quad \blacksquare} \cdot \binom{n-\delta-k-2l}{k \quad l \quad \blacksquare}$$

$$\left\langle K; V_+^{\otimes m-2i+k-3j-\delta} \otimes V_-^{\otimes n+i-2k-3l-\delta} \right\rangle,$$

where $$\binom{a+b+c}{a \quad b \quad c} = \binom{a+b+c}{a \quad b \quad \blacksquare} \stackrel{\text{def}}{=} \frac{(a+b+c)!}{a!b!c!}.$$

$m = 3, n = 2$

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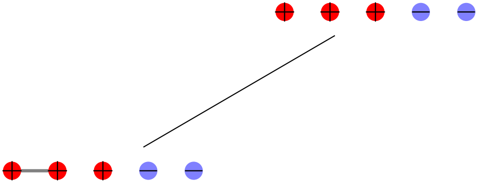

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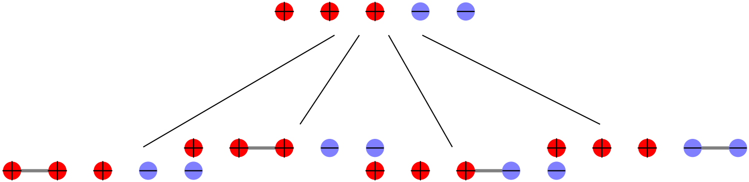

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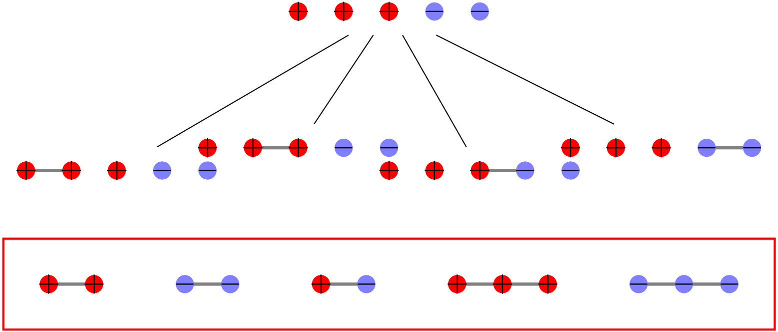

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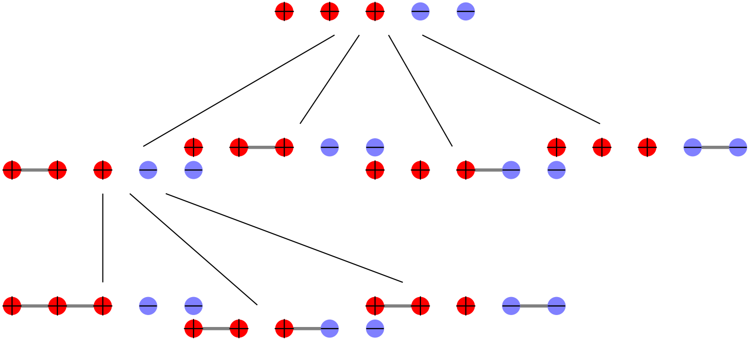

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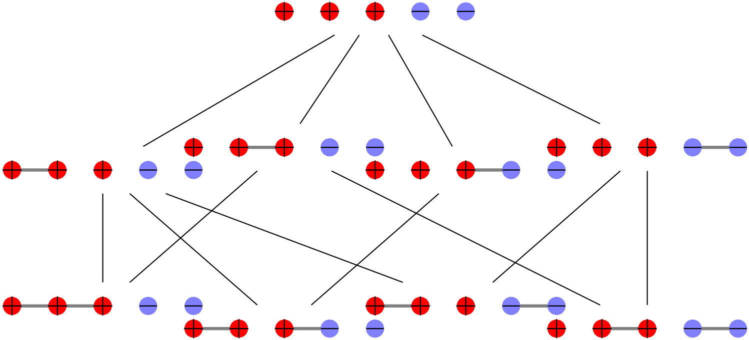

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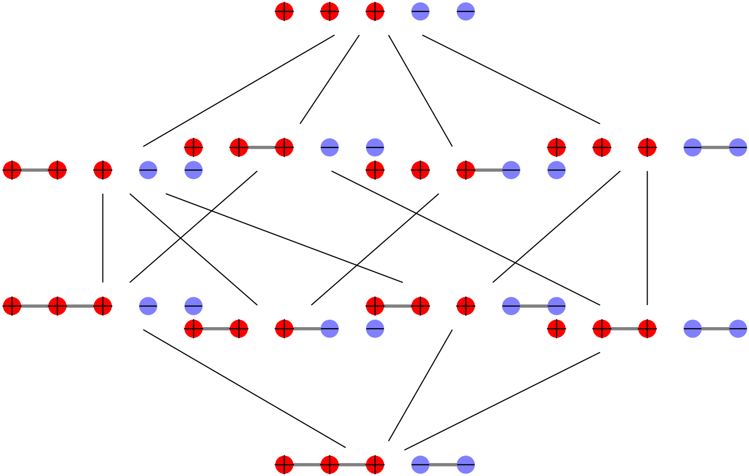

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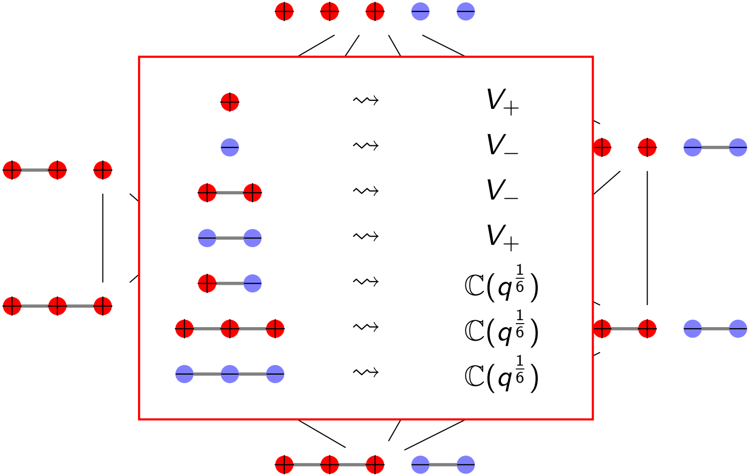

$\rightsquigarrow$ V_+
$\rightsquigarrow$ V_-
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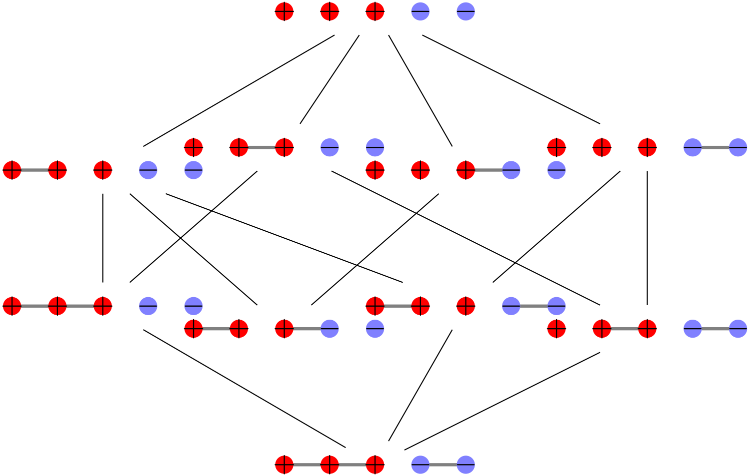

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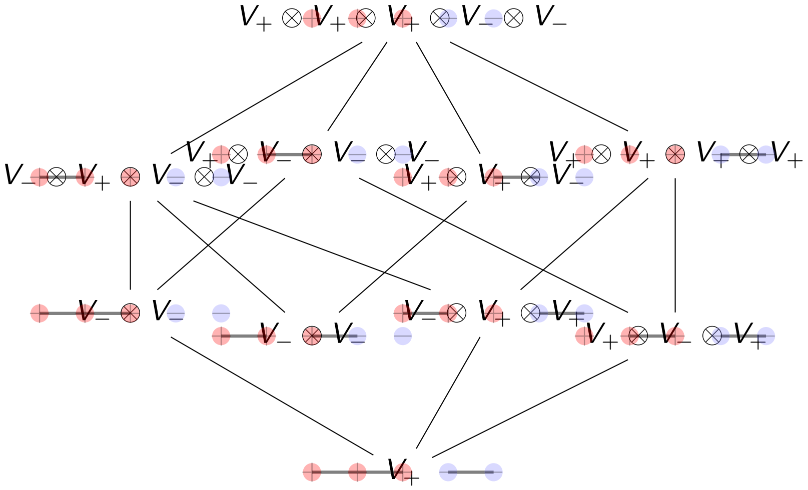

$V_+ \otimes V_+ \otimes V_+ \otimes V_- \otimes V_-$
$V_- \otimes V_+ \otimes V_- \otimes V_-$
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$V_- \otimes V_-$
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$V_- \otimes V_+ \otimes V_+$
$V_+ \otimes V_- \otimes V_+$
V_+

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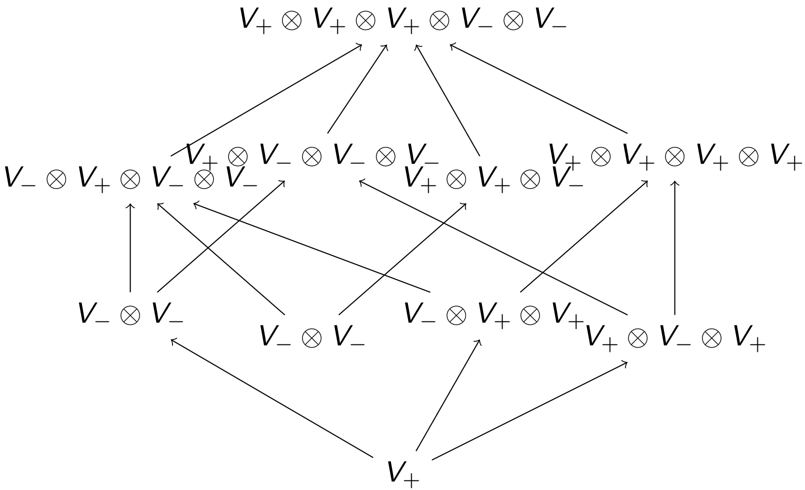
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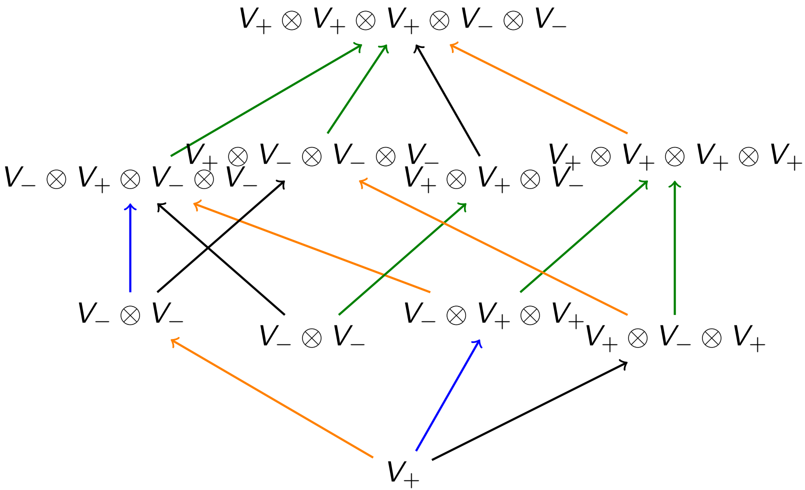

$V_+ \otimes V_+ \otimes V_+ \otimes V_- \otimes V_-$
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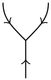
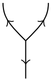
V_+

$$V_+ \simeq V_- \wedge V_- \to V_- \otimes V_-$$



$$V_- \simeq V_+ \wedge V_+ \to V_+ \otimes V_+$$



$$\mathbb{C}(q^{\frac{1}{6}}) \to V_+ \otimes V_+^* \simeq V_+ \otimes V_-$$

$$\mathbb{C}(q^{\frac{1}{6}}) \to V_- \otimes V_-^* \simeq V_- \otimes V_+$$

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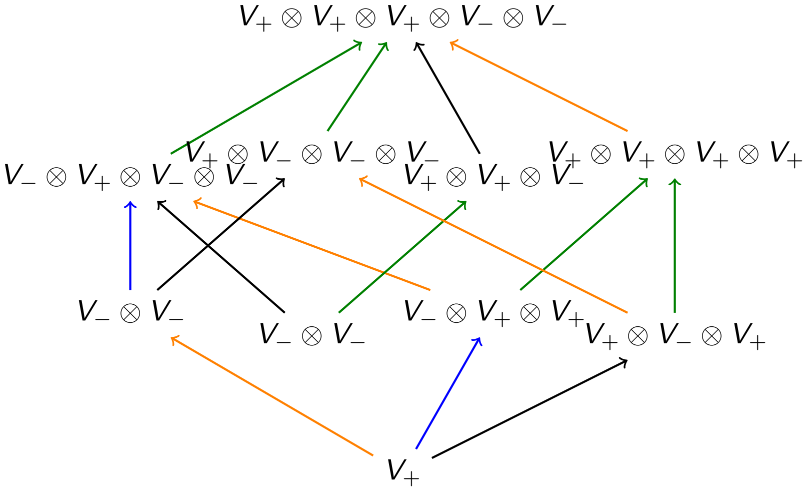

$V_+ \otimes V_+ \otimes V_+ \otimes V_- \otimes V_-$
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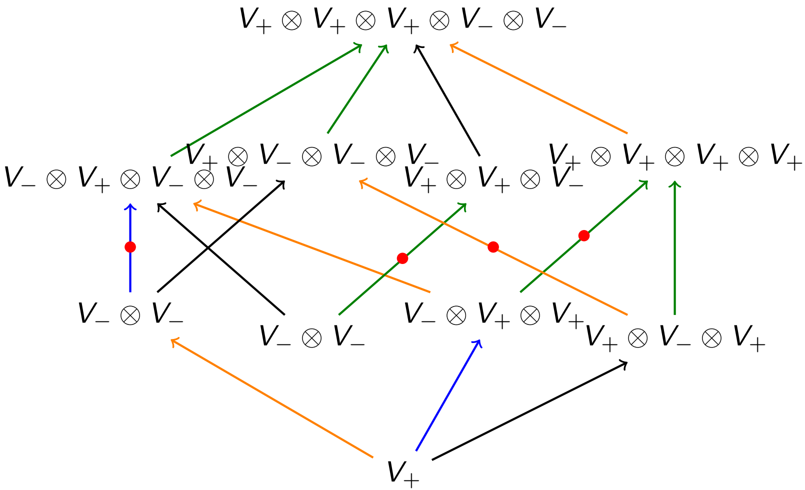

Theorem (R.), '15

For every pair of integers (m, n), the previous construction yields a resolution $C_{m,n}$ of $V_{m,n}$ in terms of tensor powers of V_+ and V_-. The isomorphism type of $C_{m,n}$ does not depend on the sign correction.

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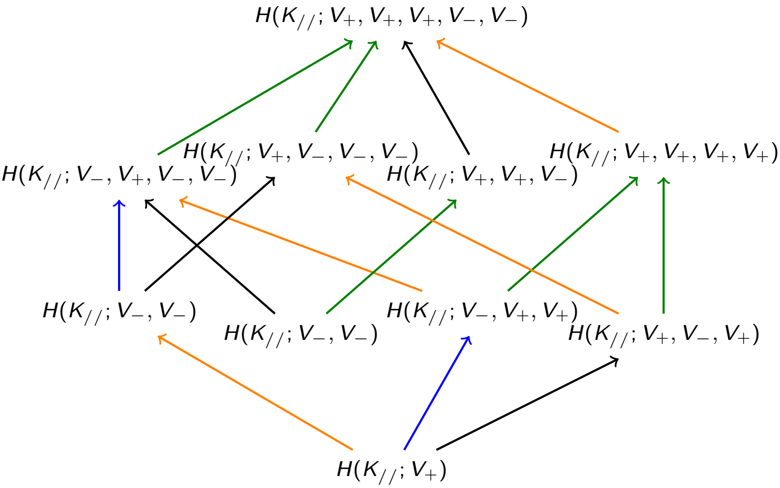
$H(K_{//}; V_+, V_+, V_+, V_-, V_-)$
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Theorem (Clark, '06)

The $\mathfrak{sl}_3$-homology is functorial.

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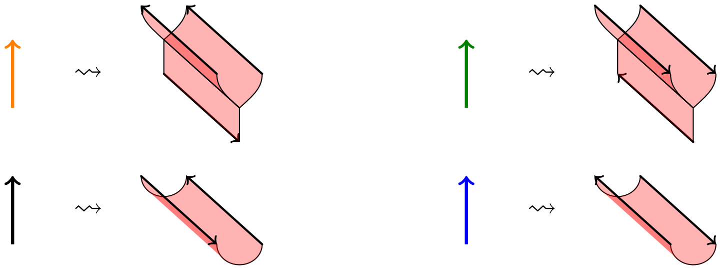
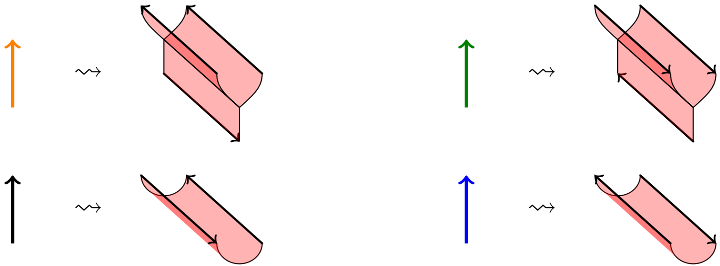


Theorem (Clark, '06)

The $\mathfrak{sl}_3$-homology is functorial. NOT for foams.



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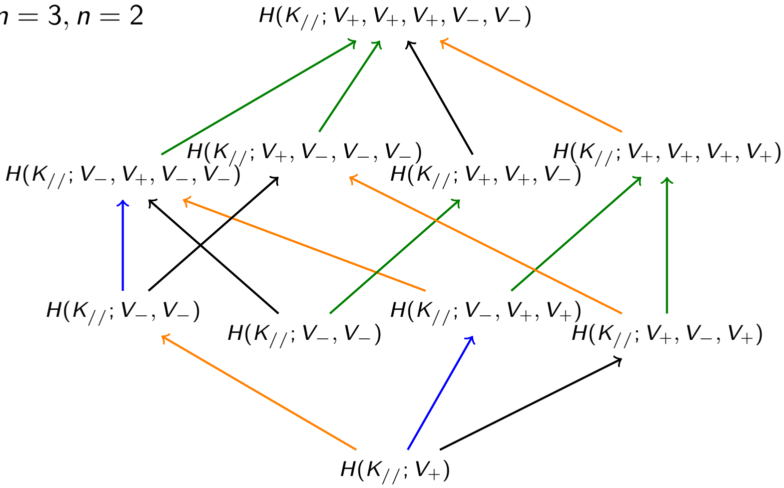

Proposition

This tri-graded complex is a framed knot invariant and categorifies the colored $\mathfrak{sl}_3$-polynomial.

Thank you!