Growth diagrams and a Lascoux's non-symmetric Cauchy identity over near staircases

Aram Emami Dashtaki
Joint work with Olga Azenhas

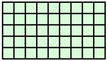
11 June 2015

Plan

- Symmetric and non-symmetric Cauchy identities.
- A non-symmetric Cauchy identity for the near staircases.

Symmetric Cauchy kernels

- Rectangle shape $\displaystyle\prod_{(i,j)\in} (1 - x_i y_j)^{-1}$



symmetric in the variables x_i and y_j separately.

It is well known in symmetric functions that symmetric Cauchy kernels can be expanded using pairs of dual bases. The Hall inner product makes the Schur functions orthonormal, or equivalently, makes the monomial symmetric functions m_λ dual to the complete homogeneous symmetric functions h_μ: $< m_\lambda, h_\mu >=< s_\lambda, s_\mu >= \delta_{\lambda,\mu}$.

It is also well understood combinatorially via RSK like bijections.

Cauchy identity

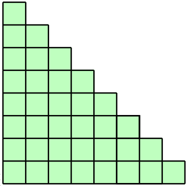
$$\prod_{(i,j)\in[n]\times[m]} (1 - x_i y_j)^{-1} = \sum_{\ell(\lambda)\leq \min\{n,m\}} s_\lambda(x) s_\lambda(y).$$

$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_m)$.

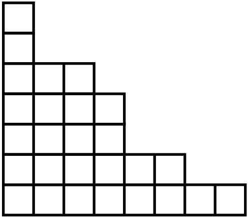
D. E. Knuth. Permutations, matrices and generalized Young tableaux. Pacific J. Math, 1970.

Non-symmetric Cauchy kernels (Lascoux, 2003)

Staircase $\prod_{i,j \in} (1 - x_i y_j)^{-1}$



Ferrers shape



Demazure operators

Demazure operators (isobaric divided differences) in type A.

$\pi_i,\ \widehat{\pi}_i : \mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[x_1, \dots, x_n]$

$$\pi_i : f \mapsto \pi_i(f) := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}, \quad 1 \leq i < n, \ \ \widehat{\pi}_i := \pi_i - 1.$$

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Let $\sigma \in \mathfrak{S}_n$. Define $\pi_\sigma = \pi_{i_1}\pi_{i_2} \ldots \pi_{i_k}$, and $\widehat{\pi}_\sigma = \widehat{\pi}_{i_1}\widehat{\pi}_{i_2} \ldots \widehat{\pi}_{i_k}$, where $s_{i_1} \ldots s_{i_k}$ is a reduced decomposition of σ.

Given the partition λ and $\alpha \in \mathbb{N}^n$ a rearrangement of λ, let $\sigma \in \mathfrak{S}_n$ be the shortest permutation (with respect to the strong Bruhat order) such that $\sigma\lambda = \alpha$. Then
Demazure character/ key polynomial

$$\kappa_\alpha(x) = \pi_\sigma(x^\lambda).$$

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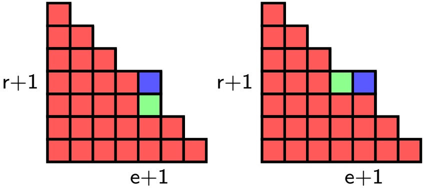
The Demazure atom

$$\widehat{\kappa}_\alpha(x) = \widehat{\pi}_\sigma(x^\lambda).$$

Demazure operators reproduce cells

- Divided differences or Demazure operators π_i^x and π_j^y increase the number of poles in the rational function $(1 - x_i y_j)^{-1}$:

$$\pi_i^x (1 - x_i y_j)^{-1} = (1 - x_i y_j)^{-1} (1 - x_{i+1} y_j)^{-1}$$
$$\pi_j^y (1 - x_i y_j)^{-1} = (1 - x_i y_j)^{-1} (1 - x_i y_{j+1})^{-1}$$

- 

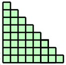
,

- $$\pi_r^x F_\rho = (\pi_r (1 - x_r y_{e+1})^{-1}) F_\eta = (1 - x_r y_{e+1})^{-1} (1 - x_{r+1} y_{e+1})^{-1} F_\eta$$
$$= F_\rho (1 - x_{r+1} y_{e+1})^{-1} = F_\lambda.$$
$$\pi_e^y F_\rho = F_\rho (1 - x_{r+1} y_{e+1})^{-1} = F_\lambda.$$

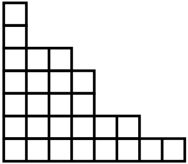
Non-symmetric Cauchy kernel expansions

Non-symmetric Cauchy identity for staircases   A. Lascoux (2003); Amy M. Fu, Alain Lascoux (2009)

$$\prod_{i,j\in \square} (1 - x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$



Lascoux's Cauchy kernel expansion over a Ferrers shape.



$$F_\lambda(X,Y) := \prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^k} (\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\nu(x))(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\nu}(y)).$$

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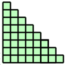

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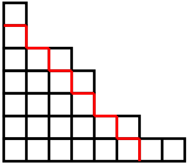

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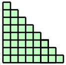

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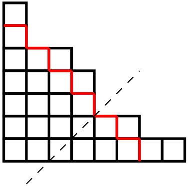

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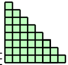

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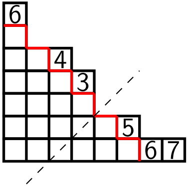

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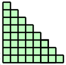

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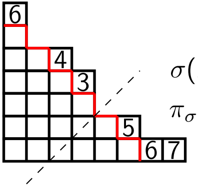

$\sigma(\lambda, NW) = s_3 s_4 s_6$ $\sigma(\lambda, SE) = s_5 s_7 s_6$

$\pi_{\sigma(\lambda,NW)} = \pi_3\pi_4\pi_6$ $\pi_{\sigma(\lambda,SE)} = \pi_5\pi_7\pi_6$

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Non-symmetric Cauchy kernel expansions

Similarly to symmetric functions Lascoux has defined an inner product with respect to which key polynomials κ_ν and $\widehat{\kappa}_\mu$ are dual $< \kappa_\nu, \widehat{\kappa}_\mu >= \delta_{\omega\nu,\mu}$. Bogdan Ion has shown that key polynomials can be obtained as a limit case of nonsymmetric Macdonald polynomials. This scalar product comes from the theory of Macdonald polynomials and in particular is a degenerating Cherednik's scalar product.

However non-symmetric Cauchy kernel expansions are not combinatorially so well understood. This difficulty comes from the fact that SSYTs are not enough to characterize key polynomials.

Expansions combinatorially established
Lascoux (2003) for stair cases via double crystal graphs.
E. and Azenhas (2014) for truncated staircases in the spirit of RSK.

Combinatorial structure of key polynomials

- Combinatorial rules for monomial expansions of the linear bases $\{\kappa_\alpha : \alpha \in \mathbb{N}^n\}$ and $\{\widehat{\kappa}_\alpha : \alpha \in \mathbb{N}^n\}$ in $\mathbb{Z}[x_1, \ldots, x_n]$.
  - Lascoux-Schützenberger (late 80's)

$$SSYT_n(\lambda) = \biguplus_{\alpha \in \mathfrak{S}_n \lambda} \{T \in SSYT_n : K_+(T) = key(\alpha)\}$$

$$key(1,0,4,0,2) = \begin{matrix} 5 & & & \\ 3 & 5 & & \\ 1 & 3 & 3 & 3 \end{matrix}$$

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  - Kashiwara crystal bases (early 90's); Haglund, Haiman, Loehr (2005); Mason (2009)

$$\hat{\kappa}_\alpha(x) = \sum_{T \in \widehat{\mathfrak{B}}_\alpha} x^T = \sum_{K_+(T) = key(\alpha)} x^T = \sum_{sh(F) = \alpha} x^F,$$

$$\kappa_\alpha(x) = \sum_{T \in \mathfrak{B}_\alpha} x^T = \sum_{K_+(T) \leq key(\alpha)} x^T = \sum_{sh(F) \leq \alpha} x^F.$$

Semi-skyline augmented fillings (SSAF) Haglund-Haiman-Loehr (2005), Mason (2009)

$$\widetilde{P} = \begin{matrix} 1 & & \\ 3 & & \\ 4 & 1 & \\ 5 & 2 & \\ 7 & 3 & 2 \end{matrix}$$

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$$\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & & 3 & 4 & 5 & & 7 \\ \hline \end{array}$$

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

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1		3	4	5		7
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$R_1 = \{7, 5, 4, 3, 1\}$ $R_2 = \{3, 2, 1\}$

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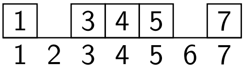

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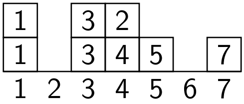

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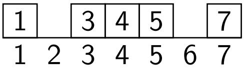

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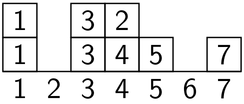

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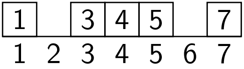

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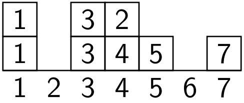

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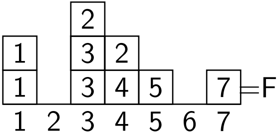

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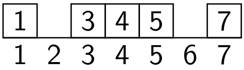

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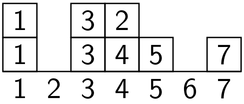

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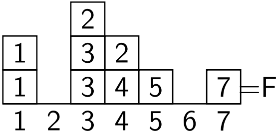
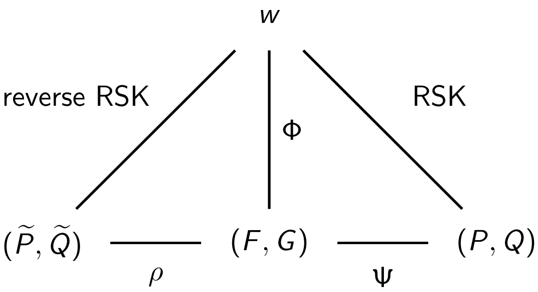


$R_3 = \{2\}$

$sh(\widetilde{P}) = (3, 2, 2, 1, 1, 0, 0)$ $\qquad$ $sh(F) = (2, 0, 3, 2, 1, 0, 1)$

SSAFs encode SSYTs with their right keys. Mason's RSK analogue detects the keys in the RSK

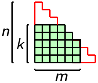


$$sh(F)^+ = sh(G)^+ = sh(P) = sh(Q) = sh(\widetilde{P}) = sh(\widetilde{Q})$$

$$key(sh(F)) = K_+(P), \; key(sh(G)) = K_+(Q)$$

$$c(F) = c(P) = c(\widetilde{P}), \; c(G) = c(Q) = c(\widetilde{Q})$$

Mason's RSK analogue restricted to the truncated staircases



$$\left\{\text{multisets of green cells in } \begin{smallmatrix} 1 \le m, k \le n \\ n+1 \le m+k \end{smallmatrix}\right\} \rightarrow \biguplus_{\nu \in \mathbb{N}^k} \left\{ (F,G){:}sh(F)=\nu,\ (sh(G),0^{n-m}) \le (0^{n-k},\omega\nu) \right\}$$

$$\begin{pmatrix} b_1 & \cdots & b_r \\ a_1 & \cdots & a_r \end{pmatrix} \rightarrow (F,G)$$

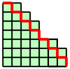
$$\prod_{\substack{(i,j)\in\lambda \\ k\le m}} (1-x_i y_j)^{-1} = \sum_{\nu\in\mathbb{N}^k} \sum_{\substack{F,G\in SSAF_n \\ sh(G)=\beta\in\mathbb{N}^m,\ sh(F)=\nu \\ (\beta,0^{n-m})\le(0^{n-k},\omega\nu)}} x^F y^G$$

$$= \sum_{\nu\in\mathbb{N}^k} \widehat{\kappa}_\nu(x)\kappa_{(0^{m-k},\alpha)}(y) \qquad E.A.(2014)$$

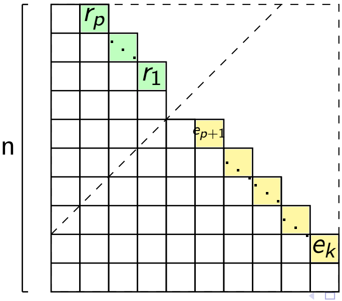
$$= \sum_{\nu\in\mathbb{N}^k} \widehat{\kappa}_\nu(x)\pi_{\sigma(\lambda,SE)}\kappa_{\omega\nu}(y). \qquad Lascoux(2003)$$

A combinatorial Cauchy kernel expansion over a near staircase

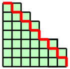
$$\prod_{(i,j)\in \square} (1 - x_i y_j)^{-1} = \sum_{\nu \in \mathbb{N}^n} \pi_{r_1} \dots \pi_{r_p} \widehat{\kappa}_\nu(x) \pi_{e_{p+1}} \dots \pi_{e_k} \kappa_{\omega\nu}(y)$$



$(Lascoux, 2003)$



Aim

Characterize the biwords fitting  via a combinatorial interpretation of the reproduction of cells above the staircase. This will be accomplished through the formulation of Mason's RSK analogue in terms of growth diagrams, and an interpretation of Demazure operators in terms of crystal or coplactic operators.

Biwords and 0-1 fillings

Representation of biword w in the $n \times n$ square.

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

1						
		1				
			1	1		
1	1					
			1			2

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1						
		1				
			1	1		
1	1					
			1			2

	×								
			×						
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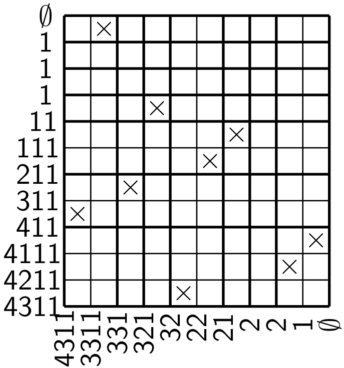
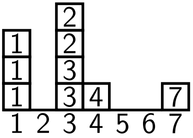
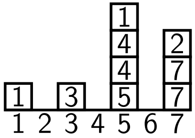
Fomin's growth diagram for RSK analogue



$$\widetilde{P} = \begin{array}{llll} 1 & & & \\ 3 & & & \\ 4 & 1 & 1 & \\ 7 & 3 & 2 & 2 \end{array}$$

$$\widetilde{Q} = \begin{array}{llll} 1 & & & \\ 3 & & & \\ 5 & 4 & 2 & \\ 7 & 7 & 4 & 1 \end{array}$$

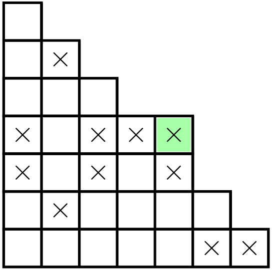
F= 

 =G

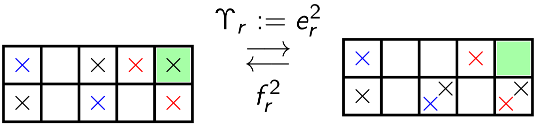
Representation of a biword in a Ferrers shape

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & \color{green}{5} & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 4 & 4 & 3 & \color{green}{4} & 1 & 1 \end{pmatrix}.$$

We represent the biword w in the Ferrers shape $\lambda = (7, 6, 5, 5, 3, 2, 1)$ by putting a cross $\times$ in the cell (i, j) of λ if $\begin{pmatrix} \mathrm{j} \\ \mathrm{i} \end{pmatrix}$ is a biletter of w.

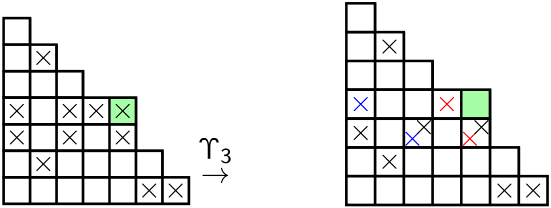


$\lambda = (7, 6, 5, 5, 3, 2, 1)$

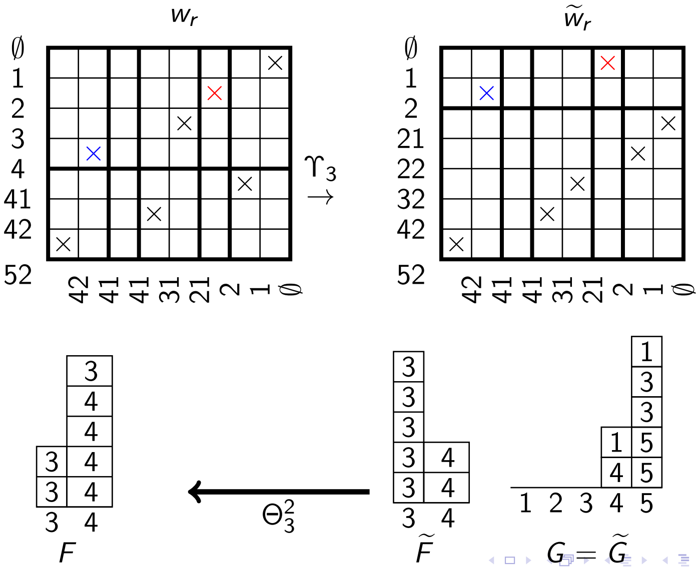


$$\begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & {\color{blue}4} & {\color{blue}3} & 4 & {\color{red}4} & {\color{red}3} & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & {\color{blue}4} & {\color{blue}3} & 3 & {\color{red}4} & {\color{red}3} & 3 \end{pmatrix}$$

The biword $\widetilde{w}$ is obtained from the biword w by applying the crystal operator e_r as long as it is possible to the second row of the biword w.



1 1 2 2 3 3 4 5 5 6 7 $\longrightarrow$ 1 1 2 2 3 3 4 5 5 6 7


w_r
$\widetilde{w}_r$
Υ_3
Θ_3^2
F
$\widetilde{F}$
$G = \widetilde{G}$

The SSAF detects the effect of the action of a crystal operator on a key

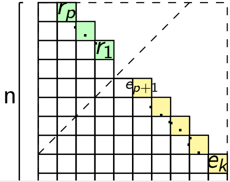
Theorem (E.)

Let λ be a Ferrers shape and w a biword consisting of a multiset of cells of λ containing the cell $(r+1, \lambda_{r+1})$ with multiplicity at least one. Let $\Phi(w) = (F, G)$ where $sh(F) = \nu$ and $sh(G) = \beta$. The following holds
(a) $\Phi(\Upsilon_r w) = (\Upsilon_r F, G)$.
(b) If $\lambda_r = \lambda_{r+1} > \lambda_{r+2} \geq 0$, for some $r \geq 1$, then $\nu_r < \nu_{r+1}$ and $sh(\Upsilon_r F) = s_r \nu$. Moreover, $\Upsilon_r w$ fits the Ferrers shape λ with the cell $(r+1, \lambda_{r+1})$ deleted.

Mason's RSK analogue restricted to near staircases

Theorem:(A., E.) Let w be a biword in lexicographic order on the alphabet $[n]$, and let $\Phi(w) = (F, G) \in SSAF_n^2$, with $sh(F) = \nu$ and $sh(G) = \beta$. Let $0 \le p \le k < n$, $1 \le r_1 < \cdots < r_p < n$, $1 \le e_{p+1} < \cdots < e_k < n$ and $r_1 + e_{p+1} > n + 1$. For each biletter $\binom{i}{j}$ in w one has $i + j \le n + 1$ except for the biletters $\binom{n - r_1 + 1}{r_1 + 1}, \ldots, \binom{n - r_p + 1}{r_p + 1}$, and $\binom{e_{p+1} + 1}{n - e_{p+1} + 1}, \ldots, \binom{e_k + 1}{n - e_k + 1}$ if and only if

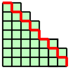
(a) $s_{e_k} \cdots s_{e_{p+2}} s_{e_{p+1}} \beta \le \omega s_{r_p} \cdots s_{r_2} s_{r_1} \nu$.
(b) $s_{e_k} \cdots s_{e_{p+2}} s_{e_{p+1}} \beta \not\le \omega s_{r_p} \cdots \widehat{s}_{r_i} \cdots s_{r_2} s_{r_1} \nu$, for $i = 1, 2, \ldots, p$.
(c) $s_{e_k} \cdots \widehat{s}_{e_i} \cdots s_{e_{p+2}} s_{e_{p+1}} \beta \not\le \omega s_{r_p} \cdots s_{r_2} s_{r_1} \nu$, for $i = p + 1, \ldots, k$.



Combinatorial expansion of the Cauchy kernel over near staircases

Theorem (A., E.) Let $0 \leq p \leq k < n$. Let $k-p < r_1 < r_2 < \cdots < r_p < n$ and $p < e_{p+1} < \cdots < e_k < n$. Then

$$\prod_{(i,j)\in} (1-x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \pi_{r_1}\ldots\pi_{r_p}\widehat{\kappa}_\nu(x)\pi_{e_{p+1}}\ldots\pi_{e_k}\kappa_{\omega\nu}(y)$$



$$= \sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p}} \sum_{(H_z,M_t)} \sum_{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}} x^F y^G,$$

where $(H_z,M_t) \in \binom{[p]}{z} \times \binom{[p+1,k]}{t}$,

$$\mathcal{A}_{z,t}^{H_z,M_t} = \left\{ (F,G)\in SSAF_n^2 : \begin{array}{c} s_{e_{j_t}}\cdots s_{e_{j_1}} sh(G) \nleq \omega s_{r_{i_z}}\cdots \widehat{s}_{r_{i_m}}\cdots s_{r_{i_1}} sh(F),\ m=1,2,\ldots,z \\ s_{e_{j_t}}\cdots \widehat{s}_{e_{j_l}}\cdots s_{e_{j_1}} sh(G) \nleq \omega s_{r_{i_z}}\cdots s_{r_{i_1}} sh(F),\ l=1,2,\ldots,t \\ s_{e_{j_t}}\cdots s_{e_{j_1}} sh(G) \leq \omega s_{r_{i_z}}\cdots s_{r_{i_1}} sh(F) \end{array} \right\}.$$

THANK YOU