Wandering Fatou Components in Dimension Two

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(Joint work with M. Astorg, X. Buff, R. Dujardin and H. Peters)

AMS-EMS-SPM Meeting – Complex Dynamics and Foliations

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Example

$X = \mathbb{P}^1(\mathbb{C})$, and $F(z) = z^2$.

- $\mathcal{F}(F) = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{S}^1$
- Fatou components: $D(0,1)$ and $\mathbb{P}^1(\mathbb{C}) \setminus \overline{D(0,1)}$

Fatou Components

Let $X = \mathbb{P}^1(\mathbb{C})$ and $F\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational function.

Theorem (Fatou, Julia, Siegel, Herman...)

A periodic Fatou component for $F\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is:

- *either the basin of a (super)attracting point,*
- *or a parabolic basin,*
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What happens when we drop one of the hypotheses?

X non-compact

Theorem (Baker 1976)

There exists an entire function $F : \mathbb{C} \to \mathbb{C}$ with a wandering Fatou component.

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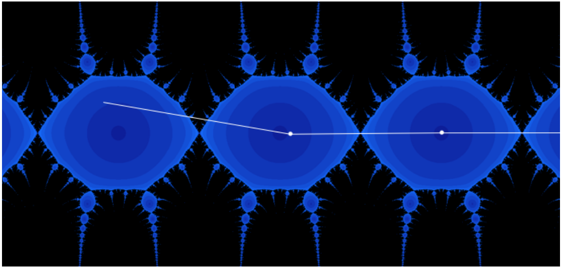
There exists an entire *function* $F : \mathbb{C} \to \mathbb{C}$ *with a wandering Fatou component.*

Example (Herman 1984 - Sullivan 1985)

The functions $z \mapsto z - e^z + 2\pi\mathrm{i}$ and $z \mapsto z + 2\pi - \sin(z)$ have wandering domains.



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In all these examples the wandering domain is not bounded.

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Idea: (Lyubich-Peters) to use skew-products, $(z, w) \mapsto (f(z, w), g(w))$.

- Lilov (2004): skew-products cannot have wandering Fatou components near a *super-attracting* invariant fiber.
- Peters and Vivas (2014): Lilov's argument is not true for *attracting* invariant fibers.

Wandering Domains

Does it exist an endomorphism of $\mathbb{P}^k(\mathbb{C})$ ($k \geq 2$) with a wandering Fatou component? Yes

Theorem (Astorg-Buff-Dujardin-Peters-R, 2014)

There exists $F\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ a holomorphic endomorphism, induced by a polynomial skew-product $F\colon \mathbb{C}^2 \to \mathbb{C}^2$, with a wandering Fatou component.

Wandering Domains

Theorem (Astorg-Buff-Dujardin-Peters-R, 2014)

The endomorphism $F\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ *defined by*

$$F(w,z) = \left(w - w^2 + w^3, z + z^2 + az^3 + \frac{\pi^2}{4} w \right)$$

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- local approach

Key proposition

- $g(w) = w - w^2 + \mathrm{O}(w^3)$ with parabolic basin $\mathcal{B}_g$,
- $f(z) = z + z^2 + \mathrm{O}(z^3)$ with parabolic basin $\mathcal{B}_f$,
- $F(w,z) = \left(g(w), f(z) + \frac{\pi^2}{4} w\right)$.

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Proposition

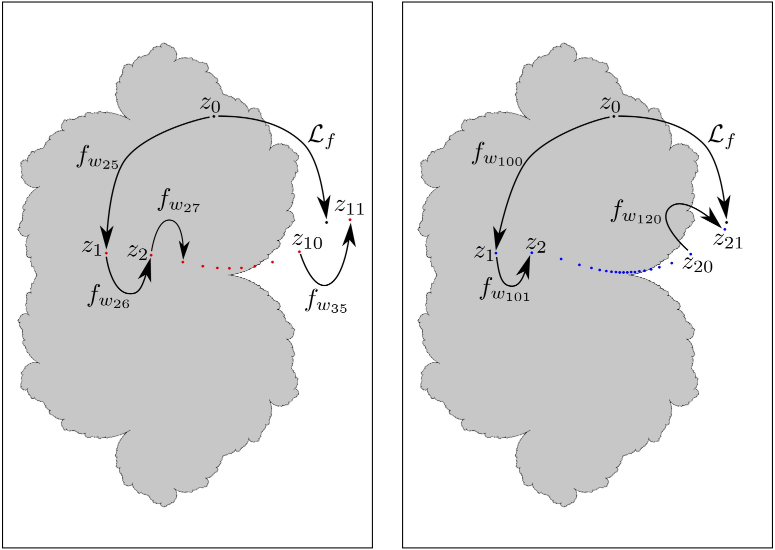
The sequence of maps

$$\mathbb{C}^2 \ni (w,z) \mapsto F^{\circ 2n+1}\left(g^{\circ n^2}(w), z\right) \in \mathbb{C}^2$$

converges locally uniformly on $\mathcal{B}_g \times \mathcal{B}_f$ *to a map*

$$\mathcal{B}_g \times \mathcal{B}_f \ni (w,z) \mapsto \left(0, \mathcal{L}_f(z)\right) \in \{0\} \times \mathbb{C}.$$

Key proposition



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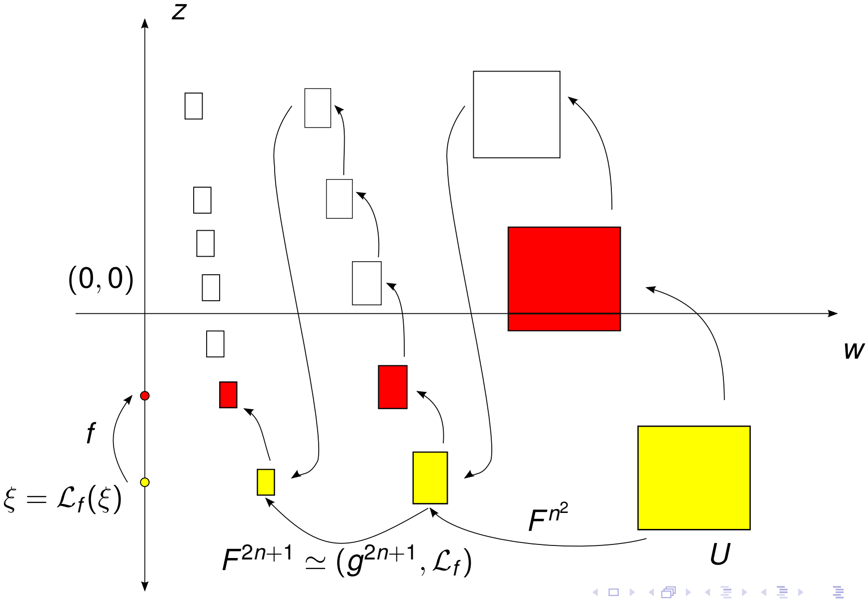
1. If $\mathcal{L}_f \colon \mathcal{B}_f \to \mathbb{C}$ has an attracting fixed point ξ, then F has a wandering domain.
2. We can choose f so that $\mathcal{L}_f \colon \mathcal{B}_f \to \mathbb{C}$ has an attracting fixed point.

Sketch of the proof



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- The sequence $(F^{\circ n^2})$ converges to
  - $(0, \xi)$ on U.
  - $(0, \xi)$ on the Fatou component Ω containing U.
  - $F^{\circ j}(0, \xi) = (0, f^{\circ j}(\xi))$ on $F^{\circ j}(\Omega)$.

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Lavaurs map

$f(z) := z + z^2 + az^3 + \mathrm{O}(z^4)$ and $T_1(Z) := Z + 1$.

Theorem (Fatou coordinates)

- *There exists* $\phi_f : \mathcal{B}_f \to \mathbb{C}$ *such that* $\phi_f \circ f = T_1 \circ \phi_f$ *and*

$$\phi_f(z) = -\frac{1}{z} - (1-a)\log\left(-\frac{1}{z}\right) + \mathrm{o}(1) \text{ if } \mathrm{Re}\left(-\frac{1}{z}\right) \to +\infty.$$

- *There exists* $\psi_f : \mathbb{C} \to \mathbb{C}$ *such that* $\psi_f \circ T_1 = f \circ \psi_f$ *and*

$$-\frac{1}{\psi_f(Z)} = Z + (1-a)\log(-Z) + \mathrm{o}(1) \text{ if } \mathrm{Re}(Z) \to -\infty.$$

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Proposition

Our limit map $\mathcal{L}_f$ *is the Lavaurs map* $\mathcal{L}_f := \psi_f \circ \phi_f$.

Parabolic Implosion

$f(z) := z + z^2 + az^3 + \mathrm{O}(z^4)$ and $f_\varepsilon(z) := f(z) + \varepsilon^2$.

Theorem (Lavaurs)

If $\dfrac{\pi}{\varepsilon_n} - 2n \to 0$ as $n \to +\infty$, then $f_{\varepsilon_n}^{\circ 2n} \to \mathcal{L}_f$ locally uniformly on $\mathcal{B}_f$.

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In our case:

$$F^{\circ 2n+1}\left(g^{\circ n^2}(w), z\right) = \left(g^{\circ n^2}(w), f_{w_{(n+1)^2-1}} \circ \cdots \circ f_{w_{n^2}}(z)\right)$$

where

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Key Proposition

For all $w \in \mathcal{B}_g$, the sequence $f_{w_{(n+1)^2-1}} \circ \cdots \circ f_{w_{n^2}}$ converges locally uniformly on $\mathcal{B}_g \times \mathcal{B}_f$ to the Lavaurs map $\mathcal{L}_f$.

Proposition 1

Let $f(z) := z + z^2 + az^3 + \mathrm{O}(z^4)$*,* $a \in \mathbb{C}$*. If* $r > 0$ *is sufficiently small and* $a \in D(1 - r, r)$ *then* $\mathcal{L}_f$ *has an attracting fixed point.*

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Proposition 2

Let $f(z) := z + z^2 + bz^4 + +\mathrm{O}(z^5)$, $b \in \mathbb{R}$. *There exists* $b \in \left(-\frac{8}{27}, 0\right)$ *such that* $\mathcal{L}_f$ *has superattracting fixed point in* $\mathcal{B}_f \cap \mathbb{R}$.

Thanks!