SPECIAL ELEMENTS OF LATTICES OF SEMIGROUP VARIETIES

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$$\forall x, y, z: \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

How special elements are defined (an example)

$$\forall y, z: \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Three equivalent form of the distributive law:

1) $\forall x, y, z: \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
2) $\forall x, y, z: \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3) $\forall x, y, z: \quad (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

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   (ii) $\forall x, z\colon \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ — y is a *costandard* element

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The modular law: $\forall x, y, z: \quad y \leq z \rightarrow (x \vee y) \wedge z = (x \wedge z) \vee y$

Three equivalent form of the distributive law:

1) $\forall x,y,z\colon \quad x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$

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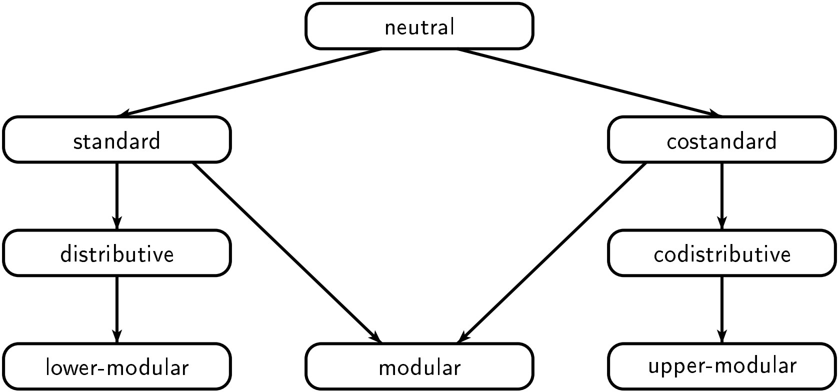
(vi) $\forall y,z\colon \quad y\le z\to(x\vee y)\wedge z=(x\wedge z)\vee y$ — x is a *modular* element

(vii) $\forall x,z\colon \quad y\le z\to(x\vee y)\wedge z=(x\wedge z)\vee y$ — y is a *lower-modular* element

(viii) $\forall x,y\colon \quad y\le z\to(x\vee y)\wedge z=(x\wedge z)\vee y$ — z is an *upper-modular* element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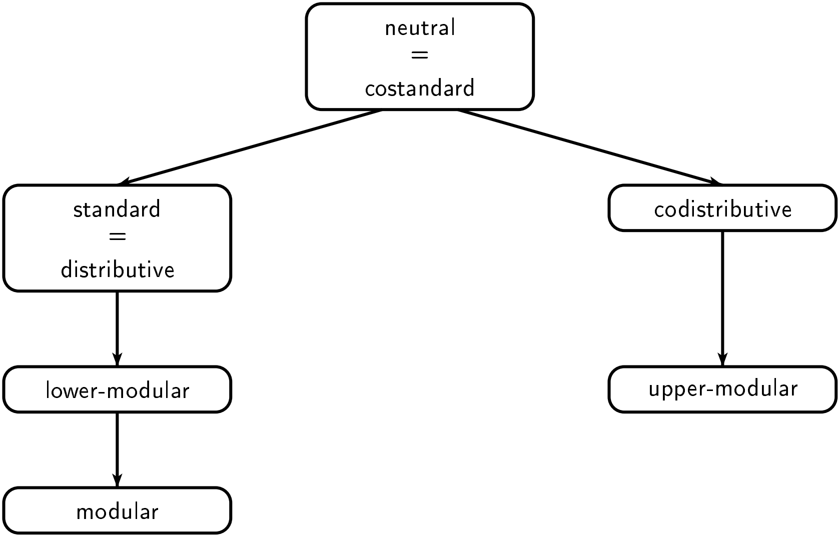
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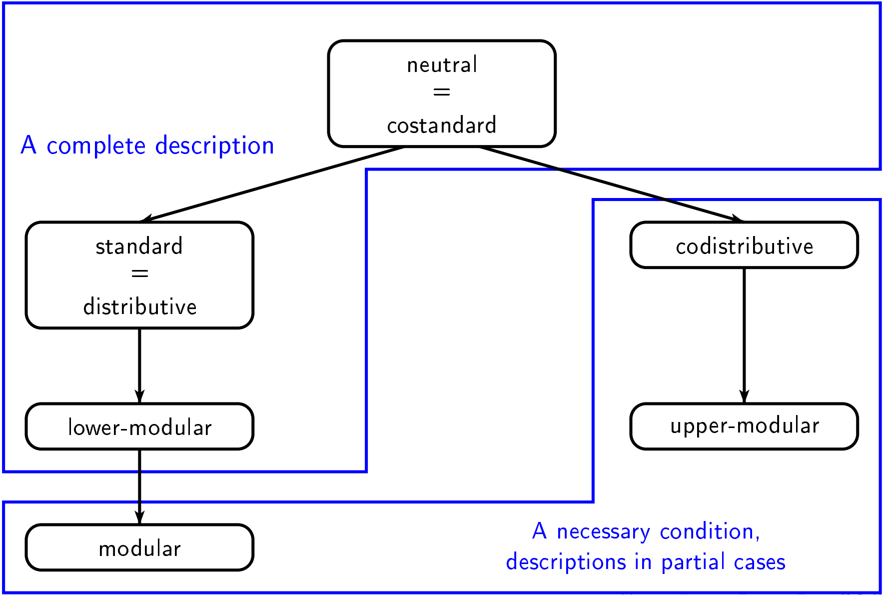
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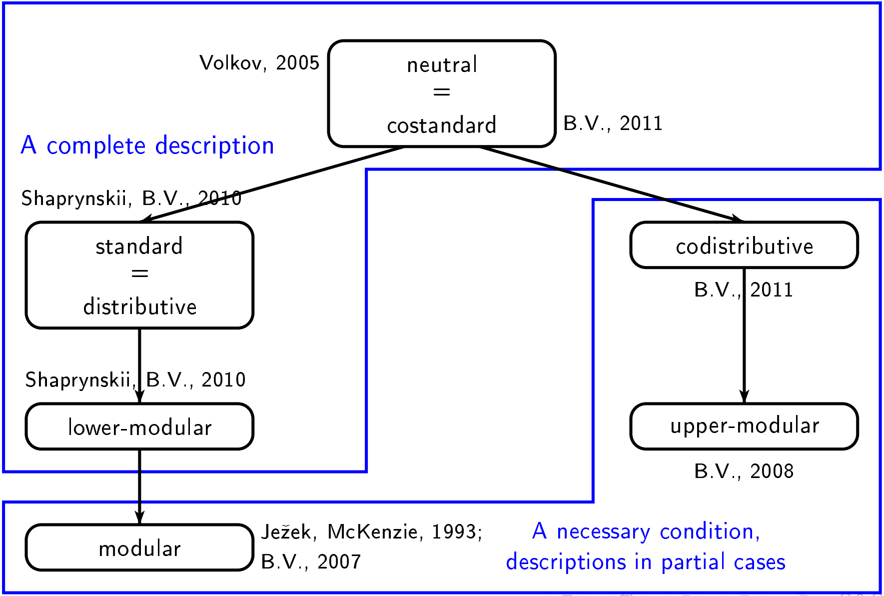
Volkov, 2005
neutral
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B.V., 2011
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Shaprynskii, B.V., 2010
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Ježek, McKenzie, 1993;
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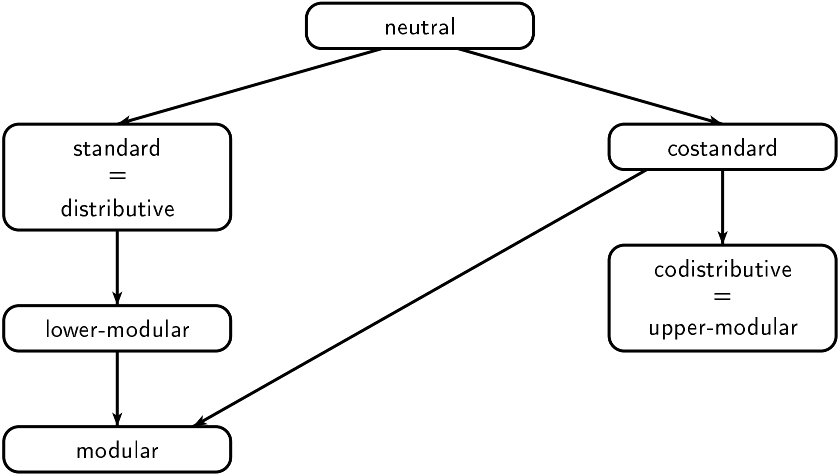
The lattice **Com** of all commutative varieties



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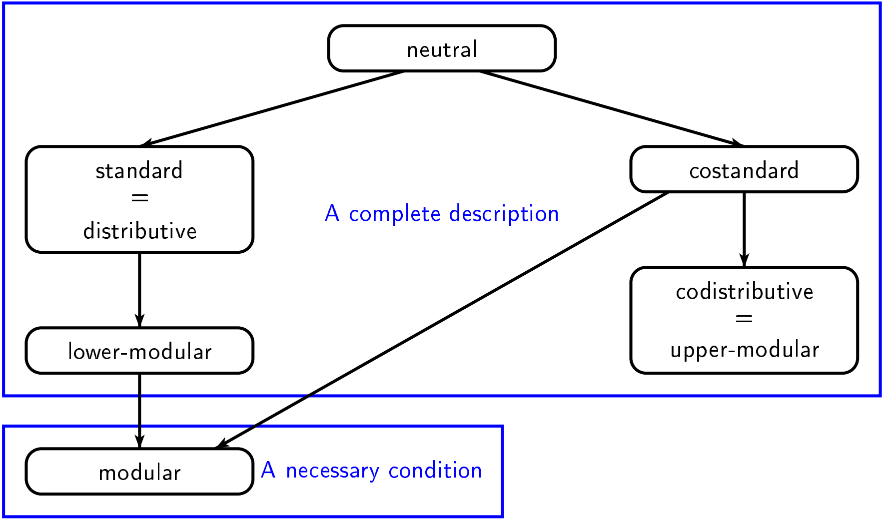

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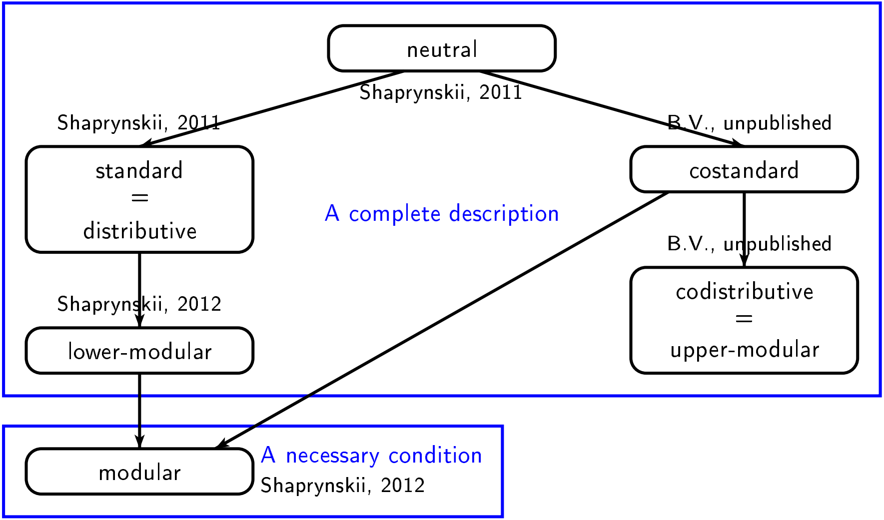

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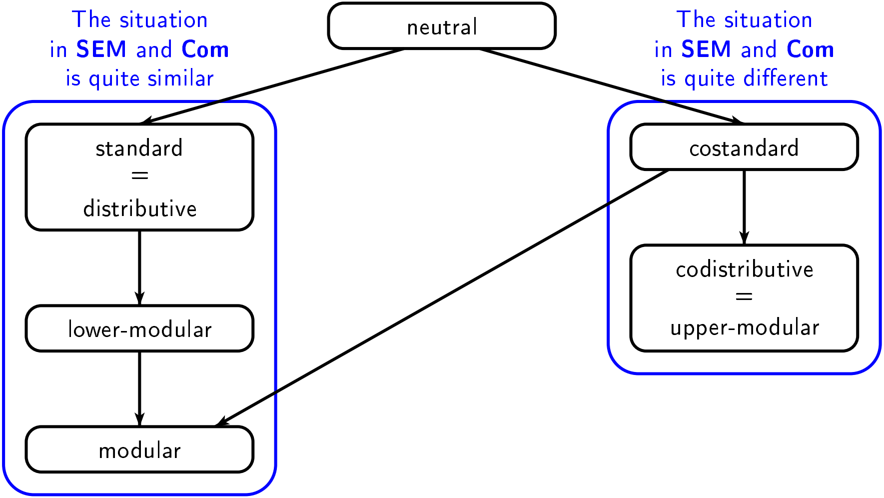

Special elements in **SEM** and **Com**: the similarities

In both the lattices **SEM** and **Com**:

- an element is distributive if and only if it is standard;
- an element is modular whenever it is lower-modular.

Results concerning modular or lower-modular or distributive (= standard) elements in these two lattices have quite similar formulations. For instance:

Lower-modular elements in SEM (Shaprynskii and B.V., 2010)

A semigroup variety $\mathcal{V}$ is a lower-modular element in **SEM** $\Longleftrightarrow$ *either $\mathcal{V}$ is the variety $\mathcal{SEM}$ of all semigroups or $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is given by identities of the form $w = 0$ only, and $\mathcal{SL}$ is the variety of semilattices.*

Lower-modular elements in Com (Shaprynskii, 2012)

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There are other pairs of analogous statements of such a kind (for instance, necessary conditions for modular elements in **SEM** and **Com**).

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Modular and lower-modular elements in some varietal lattices (Shaprynskii, 2012)

Let $\mathcal{X}$ be an overcommutative semigroup variety and $\mathcal{V}$ a periodic subvariety of $\mathcal{X}$. If $\mathcal{V}$ is either a modular or a lower-modular element of the lattice $\mathbf{L}(\mathcal{X})$ then either $\mathcal{V} = \mathcal{N}$ or $\mathcal{V} = \mathcal{SL} \vee \mathcal{N}$ where $\mathcal{N}$ is a nil-variety.

The lattices **SEM** and **Com** are lattices of the form $L(\mathcal{X})$ with two extremal values of $\mathcal{X}$, namely $\mathcal{X} = \mathcal{SEM}$ and $\mathcal{X} = \mathcal{COM}$ respectively.

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Special elements in **SEM** and **Com**: the differences

In the lattice **SEM**:

- an element is neutral if and only if it is costandard;
- the properties of being upper-modular and codistributive elements are not equivalent.

In the lattice **Com**:

- the properties of being neutral and costandard elements are not equivalent;
- an element is upper-modular if and only if it is codistributive.

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Two nilvarieties

$$\mathcal{I} = \mathrm{var}\{x^2yz = 0,\ x^2y = xy^2,\ xy = yx\}, \quad \mathcal{J} = \mathrm{var}\{x^2y = 0,\ xy = yx\}$$

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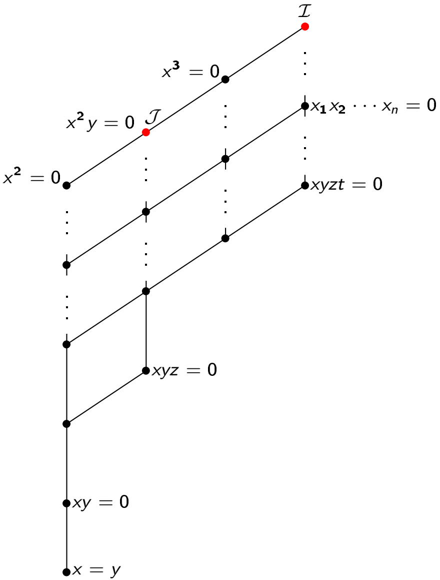

Theorem (B.V., 2015)

For a commutative semigroup variety $\mathcal{V}$, the following are equivalent:

a) *$\mathcal{V}$ is an upper-modular element of* **Com**;
b) *$\mathcal{V}$ is a codistributive element of* **Com**;
c) *one of the following holds:*
   (i) $\mathcal{V} = \mathcal{COM}$;
   (ii) $\mathcal{V} \subseteq \mathcal{A} \vee \mathcal{SL} \vee \mathcal{I}$ *where $\mathcal{A}$ is an Abelian periodic group variety;*
   (iii) $\mathcal{V} \subseteq \mathcal{C} \vee \mathcal{J}$ *where* $\mathcal{C} = \mathrm{var}\{x^2 = x^3, xy = yx\}$.

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A commutative semigroup variety $\mathcal{V}$ is a costandard element of **Com** *if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{I}$.*

For comparison: $\mathcal{V}$ is a neutral element of **Com** if and only if either $\mathcal{V} = \mathcal{COM}$ or $\mathcal{V} \subseteq \mathcal{SL} \vee \mathcal{J}$ (Shaprynskii, 2011).

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Upper modularity in Com and SEM

A commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{COM}$ is an upper-modular element of **Com** *if and only if it is an upper-modular element of* **SEM**.

For comparison: a commutative semigroup variety $\mathcal{V}$ with $\mathcal{V} \neq \mathcal{COM}$ is a neutral element of **Com** if and only if it is a modular element of **SEM** (follows from results by B.V., 2007, and Shaprynskii, 2012).

A nil-case

For a commutative nil-variety of semigroups $\mathcal{N}$, the following are equivalent:

a) $\mathcal{N}$ *is an upper-modular element of* **Com**;
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THANK YOU FOR YOUR ATTENTION!