Decidability vs Undecidability of the word problem in HNN-extensions of inverse semigroups.

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Yamamura's HNN-extension

- $[S; A_1, A_2; \varphi]$ where $S = Inv\langle X|R\rangle \simeq (X \cup X^{-1})^+/\omega$, $\varphi : A_1 \to A_2$, A_1, A_2 inverse subsemigroups of S;
- $e, f \in E(S)$ s.t. $e \in A_1 \subseteq eSe$ and $f \in A_2 \subseteq fSf$;
- $S^* = Inv\langle S, t \mid t^{-1}at = \varphi(a), t^{-1}t = f, tt^{-1} = e, \forall a \in A_1\rangle$ is called the *HNN-extension* of S associated with $\varphi : A_1 \to A_2$.
- There is another approach that extends the notion of HNN-extension from groups to inverse semigroups given by Gilbert. This HNN-extension in the sense of Gilbert embeds into the HNN-extension in the sense of Yamamura (proved by A. Yamamura in 2007).

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The word problem under "nice" conditions

- In the group case thanks to Britton's Lemma:

Theorem

Let $G^* = \langle t, G | t^{-1} a t = \varphi(a), a \in A_1 \rangle$ be an HNN-extension of a group G. If G has solvable word problem and the membership problem for A_1, A_2 is solvable, and φ, φ^{-1} are effectively calculable, then G^* has solvable word problem.

The word problem under “nice” conditions

- In the inverse semigroup case under the same conditions.

Theorem

The word problem for Yamamura's HNN-extensions S^* of inverse semigroups $[S; A_1, A_2; \varphi]$ is undecidable even if

- *S has finite $\mathcal{R}$-classes (therefore solvable word problem);*
- *the membership problem for A_1, A_2 in S is decidable, and $A_1 \simeq A_2$ is a free inverse semigroup with zero and finite rank;*
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Schützenberger automata

Definition (Schützenberger graphs, Stephen)

Let $S = Inv\langle X|R\rangle = (X \cup X^{-1})^+/\rho$ and $w \in (X \cup X^{-1})^+$ the Schützenberger graph $S\Gamma(X, R; w)$ is an inverse word graph whose vertices are the elements of the $\mathcal{R}$-class of $w\rho$ and whose edge set is $\{(\nu, x, \mu)|x \in X \cup X^{-1}, \nu(x\rho) = \mu\}$.

In other words it is the connected component of the Cayley graph of S containing $w\rho$.

- $\mathcal{A}(X, R, w) = (ww^{-1}\rho, S\Gamma(X, R; w), w\rho)$ is the Schützenberger automaton of w with respect to $\langle X|R\rangle$.
- it is a deterministic automaton
- $L[\mathcal{A}(X, R; w)] = \{v \in (X \cup X^{-1})^+|w\rho \leq v\rho\}$
- $w\rho = w'\rho$ iff $L[\mathcal{A}(X, R; w)] = L[\mathcal{A}(X, R; w')]$.

How to build a Schützenberger automaton

- From the linear automaton of w via two fundamental operations.
- Folding/determinization: fold a pair of edges labelled by the same element starting from the same vertex.
- Expansion: if v labels a path from a vertex ν to a vertex μ and $(s, t) \in R$ add a path labelled by t from ν to μ.
- Iteratively applying these operations a directed system of inverse automata is obtained

$$\mathcal{A}_1 \to \mathcal{A}_2 \to \ldots \to \mathcal{A}_i \to \ldots$$

whose directed limit is the Schützenberger automata $\mathcal{A}(X, R; w)$.

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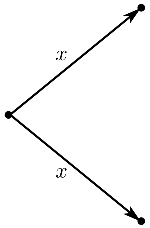

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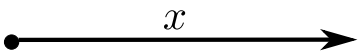

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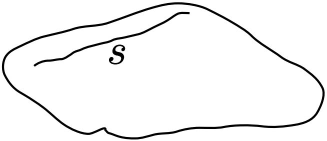

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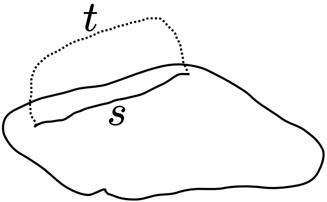

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The Shape of the Schützenberger automata for an HNN-extension

- Inverse graph Γ on $X \cup t$;
- A lobe of Γ is a maximal connected component labelled by elements of X;
- Lobe graph $G(\Gamma)$: vertices the set of lobes and two lobes are adjacent if there is a edge $p \xrightarrow{t} q$ connecting them;
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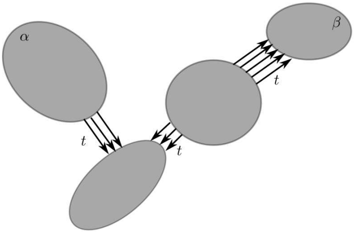
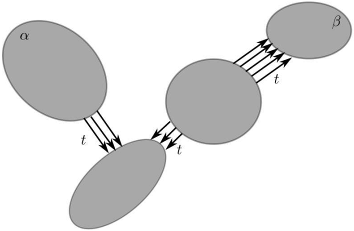


The Shape of the Schützenberger automata for an HNN-extension

Proposition (Jajcayova)

The Schützenberger automaton of the HNN-extension S^* with respect to the presentation $\langle S, t \mid t^{-1}at = \varphi(a), t^{-1}t = f, tt^{-1} = e, \forall a \in A_1 \rangle$ is a weak t-opuntoid.



Passing from the amalgamated free product

- The proof relies on a analogous undecidability result for amalgamated free products of inverse semigroups.
- Amalgam $[S_1, S_2; U, \omega_1, \omega_2]$ with $S_1 = Inv\langle X_1 | R_1 \rangle$ $S_2 = Inv\langle X_2 | R_2 \rangle$ with $X_1 \cap X_2 = \emptyset$, $\omega_i : U \hookrightarrow S_i$, $i = 1, 2$.
- The amalgamated free product
  $S_1 *_U S_2 = Inv\langle X_1 \cup X_2 | R_1 \cup R_2 \cup W \rangle$ where
  $W = \{(u\omega_1, u\omega_2) | u \in U\}$

Theorem (R., Silva)

The word problem for $S_1 *_U S_2$ of inverse semigroups may be undecidable even if we assume the following conditions.

- *S_1 and S_2 have finite $\mathcal{R}$-classes*
- *U is a free inverse semigroup with zero of finite rank*
- *the membership problem of $\omega_i(U)$ is decidable in S_i for $i = 1, 2$*
- *ω_1, ω_2 and their inverses are computable functions.*

Associate an HNN-extension to an amalgam

Theorem (Cherubini, R.)

Let $[S_1^{e_1}, S_2^{e_2}; U^1, \omega_1^1, \omega_2^1]$ *be the free product with amalgamation with adjoint identities, associate the HNN-extension* $[S_1^{e_1} * S_2^{e_2}; U_1^{e_1}, U_2^{e_2}; (\omega_1^1)^{-1} \circ \omega_2^1]$ *and* $S^* = Inv\langle \overline{X} | \overline{R} \cup R_{HNN} \rangle$*, then*

$$S^*/\rho \simeq (S_1^{e_1} *_{U^1} S_2^{e_2}) \simeq (S_1 *_U S_2)^1$$

where $(S_1 *_U S_2)^1$ *denotes* $S_1 *_U S_2$ *with adjoint identity 1 and* ρ *is the congruence on* S^* *generated by the relation* $t = e_1$, $t = e_2$.

From the point of view of Schützenberger automata

- Schützenberger automaton $\mathcal{A}(X_1 \cup X_2, R_1 \cup R_2 \cup W, w)$ of the word w with respect to $S_1 *_U S_2$ from the Schützenberger $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w')$ of the associated HNN-extension;
- Factorize $w = w_1 w_2 ... w_{2n-1} w_{2n}$, $w_1 \in (X_1 \cup X_1^{-1})^*$, $w_{2i} \in (X_2 \cup X_2^{-1})^+$, $w_{2i+1} \in (X_1 \cup X_1^{-1})^+$, $1 \leq i \leq n-1$;
- Considered the associate separated normal form

$$w' = w_1 e_1 t e_2 w_2 e_2 t^{-1} e_1 ... e_2 t^{-1} e_1 w_{2n-1} e_1 t e_2 w_{2n}$$

From the point of view of Schützenberger automata

Proposition (Cherubini, R.)

$\mathcal{A}(X_1 \cup X_2, R_1 \cup R_2 \cup W, w)$ can be obtained from $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w')$ of the separated normal form w' of w by identifying the initial and terminal vertices of each t-edge and then deleting all the obtained loops labelled by t.

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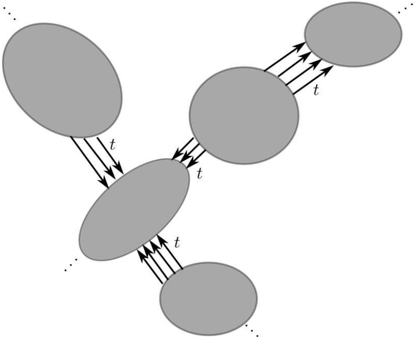

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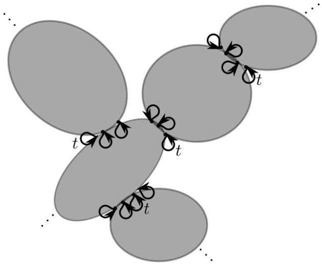

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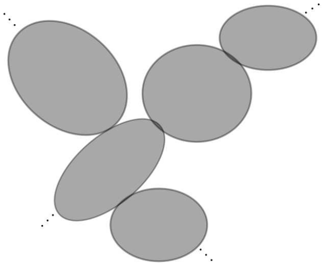

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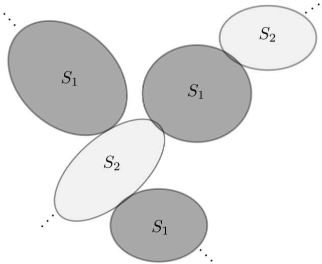
- More precisely the lobes are quotients of Schützenberger automata of either S_1 or S_2 with a tree-like structure (weak opuntoid class of inverse graphs denoted by $\mathcal{C}$).
- This means that the Schützenberger automaton of the separated normal form has a particular shape (class of separated weak t-opuntoid inverse graphs denoted by $\mathcal{C}_t$)



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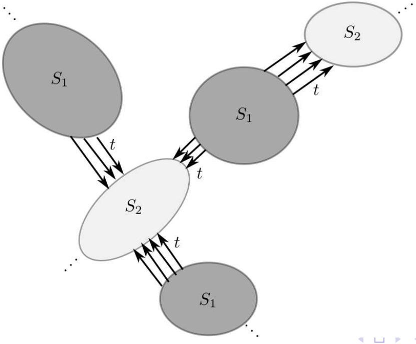

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The map $\psi : \mathcal{C}_t \to \mathcal{C}$, which is defined by identifying the initial vertex with the terminal vertex of each t-edge and then erasing the formed loops, is a bijection.

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Proposition

Let $w_1, w_2 \in (X_1 \cup X_2 \cup X_1^{-1} \cup X_2^{-1})^+$, and let w_1' and w_2' be their corresponding separated normal forms.
Let $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_1') = (\alpha, \Gamma_1, \beta)$, $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_2') = (\alpha', \Gamma_2, \beta')$ be the corresponding Schützenberger automata which are separated weak t-opuntoid automata with the property that:

$$\psi\left((\alpha, \Gamma_1, \beta)\right) = \psi\left((\alpha', \Gamma_2, \beta')\right)$$

then there are $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ such that

$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{ in } S^*$$

Sketch of the proof (I)

- Consider the amalgam $[S_1; S_2, U; \omega_1, \omega_2]$ associated to this theorem.

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- Consider the amalgam $[S_1; S_2, U; \omega_1, \omega_2]$ associated to this theorem.
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Lemma

Let $w_1, w_2 \in (X_1 \cup X_2 \cup X^{-1} \cup X_2^{-1})^+$ with w_1' and w_2' their corresponding separated normal forms, respectively. Then $w_1 = w_2$ in $S_1 *_U S_2$ if and only if there are $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ such that

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- Hence, if the word problem for $[S_1^{e_1} * S_2^{e_2}; U_1^{e_1}, U_2^{e_2}; (\omega_1^1)^{-1} \circ \omega_2^1]$ would be solvable, then the word problem for $S_1 *_U S_2$ would be solvable, a contradiction.

Lemma

Let $w_1, w_2 \in (X_1 \cup X_2 \cup X^{-1} \cup X_2^{-1})^+$ with w_1' and w_2' their corresponding separated normal forms, respectively. Then $w_1 = w_2$ in $S_1 *_U S_2$ if and only if there are $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ such that

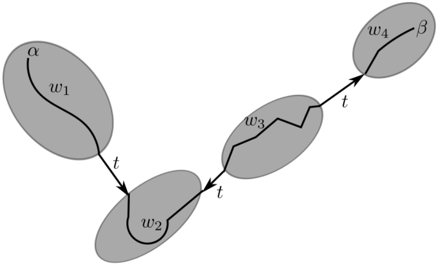
$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{ in } S^*$$

The “boundaries” decidability/undecidability

- Future work: sketch the boundary between decidability/undecidability both for HNN-extensions and free product with amalgamations.
- By the previous results, we may assume that the starting semigroups has finite $\mathcal{R}$-classes.
- We have some partial results.

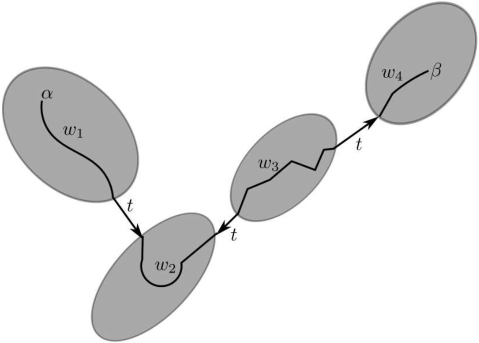
Building the Schützenberger automaton

- Starting from the linear automaton of $w_1 t w_2 t^{-1} w_3 t w_4$



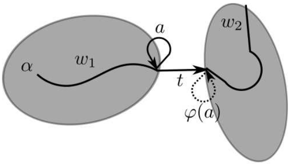
Building the Schützenberger automaton

- Close the lobes, i.e. apply all the expansions and foldings relative to S



Building the Schützenberger automaton

- $a \in E(A_1)$ labels a loop, but $\varphi(a)$ does not labels a loop. Make an expansion, then close the lobe.



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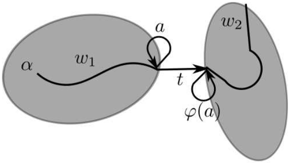

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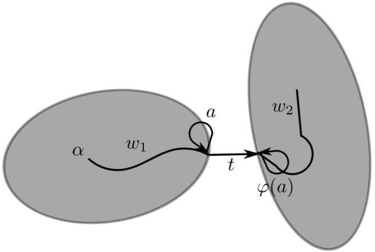
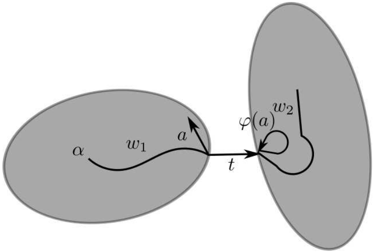


Building the Schützenberger automaton

- A loop labeled by $\varphi(a)$ for which a in not a loop in the corresponding vertex. Make an expansion: equivalent to quotient the path into a loop, then close the lobe.



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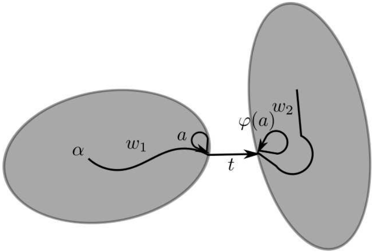
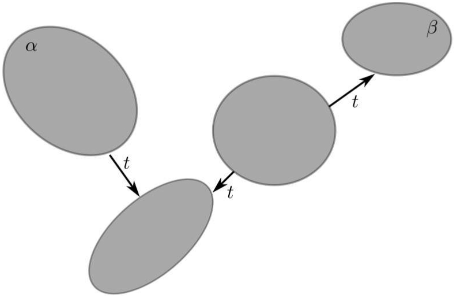


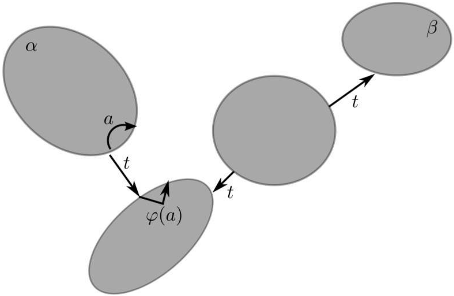
Building the Schützenberger automaton

- In the "limit" the lobe graph is finite, however, each lobe may not be finite.



Building the Schützenberger automaton

- One to one correspondence: add “t”.



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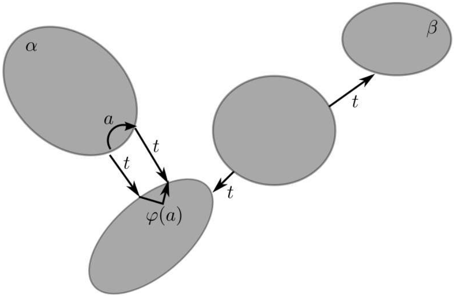
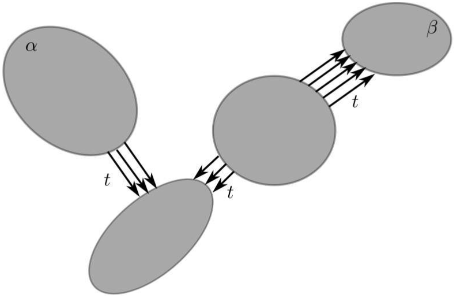


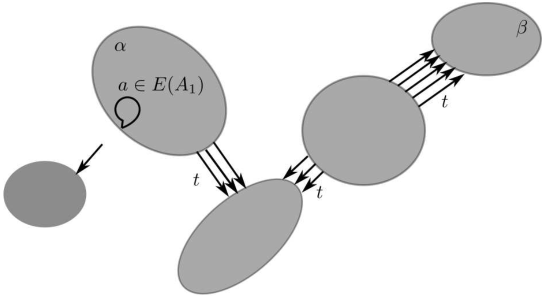
Building the Schützenberger automaton

- Finally one obtains a “graphical normal form”



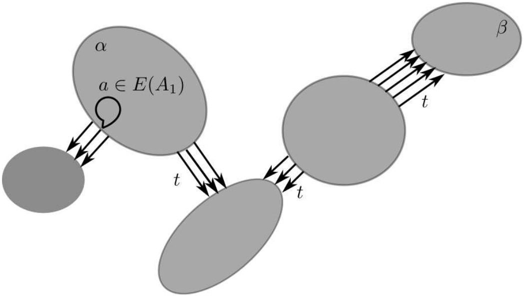
The Graphical Normal Form

- It is possible to prove that there is a way to build in the limit a new lobe, let us call it an "external lobe".



The Graphical Normal Form

- We can glue to the previous automaton and iterate this process we get at the limit the Schützenberger automaton.



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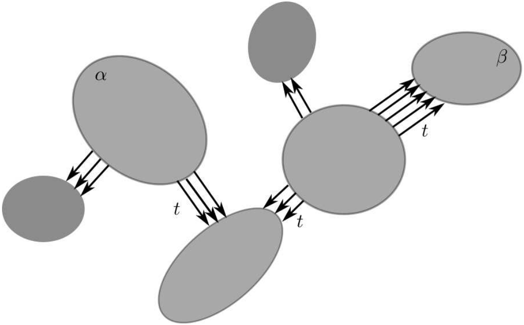

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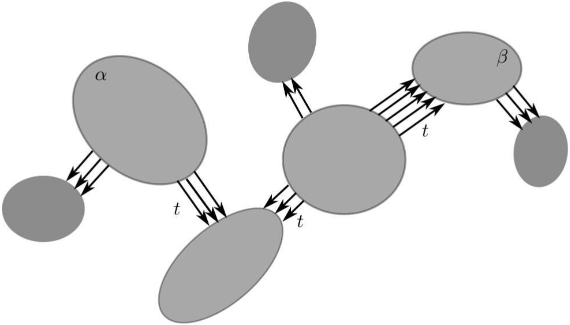

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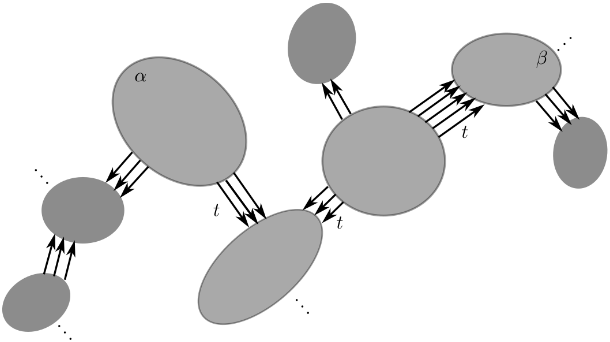

Decidability conditions

- If the lobes in the graphical normal form are finite, and the lobes that we add are finite $\Rightarrow$ the word problem is solvable!
- Main problem: the closure of a lobe (even if it is finite) is not finite.
- Minimality property: a lobe is said to satisfy the *m*-property, if it has a minimum idempotent labelling a loop at some vertex.
- A lobe having the *m*-property is finite (not true the converse). Furthermore, the closure of a lobe with the *m*-property has the *m*-property.

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- If all the lobes of the “graphical normal form” have the *m*-property, what about the added external lobes?
- This is similar to the lower-bounded condition considered by Jajkayova/Bennet conditions. It actually includes both the lower-bounded and the finite case.
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Theorem

The external lobes have the m-property if and only if for any $e \in E(S)$ we get

$$U_i(e) = \{g \in E(A_i) : g \geq e\} \neq \emptyset \Rightarrow U_i(e) \text{ has a minimum.}$$

for $i = 1, 2$.

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THANK YOU!