On the link invariants associated to the framization of knot algebras

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Knot Algebras and Link Invariants

Definition

A knot algebra is an algebra that is applied to the construction of invariants of classical links. More precisely, a knot algebra A is a trio (A, π, τ), where π is a representation of the Artin braid group in A and τ is a Markov trace defined on A.

The invariant obtained by the knot algebra A is constructed essentially from the composition $\tau \circ \pi$ after re–scaling and normalizing τ according to the braid equivalence in the given braid category.

Knot Algebras and Link Invariants

Knot algebra	Invariant
Iwahori–Hecke algebra	Homflypt polynomial
Temperley–Lieb algebra	Jones polynomial & bracket polynomial
BMW algebra	Kauffman polynomial
B–type & affine Hecke algebras	Lambropoulou invariants
Singular Hecke algebra	Kauffman–Vogel & Paris–Rabenda invariants
Rook algebra	Jones & Alexander polynomials

Table: Examples of knot algebras.

What is (modular) Framization?

- Proposed by Juyumaya & Lambropoulou.
- The addition of framing generators to a knot algebra.
- A new algebra is obtained which is related to framed knots.

The Hecke algebra of type A

$$\mathrm{H}_n(u) = \mathrm{Alg}_{\mathbb{C}}\left\{ h_1,\ldots,h_{n-1} \middle| \begin{array}{cc} h_ih_j = h_jh_i & |i-j|>1 \\ h_ih_jh_i = h_jh_ih_j & |i-j|=1 \\ h_i^2 = 1+(q-q^{-1})h_i & \end{array} \right\}$$

- $\mathrm{H}_n(q)$ can be considered as a quotient of $\mathbb{C}B_n$.

Theorem (Ocneanu 1984)

For any $\zeta \in \mathbb{C}^{\times}$ there exists a unique linear function

$$\tau : \cup_{\infty}\mathrm{H}_n(q) \longrightarrow \mathbb{C}[\zeta]$$

that can be inductively defined by the following rules:

1. $\tau(1) = 1$
2. $\tau(ab) = \tau(ba) \quad a,b \in \mathrm{H}_n(q)$
3. $\tau(ah_n) = \zeta\tau(a) \quad a \in \mathrm{H}_n(q)$

Link invariants through τ

Isotopy classes of links are in bijection with equivalence classes of braids under:

Markov Moves

I. Conjugation: $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in B_n$

II. Stabilization: $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$

Markov trace

2. Conjugation: $\tau(ab) = \tau(ba)$
3. Markov Property: $\tau(ah_n) = \zeta\tau(a)$

$a, b \in \mathrm{H}_n(q)$.

The trace τ should be rescaled so that:

$$\tau(ah_n) = \tau(ah_n^{-1}) \qquad (a \in \mathrm{H}_n(q))$$

The Homflypt polynomial

Denoting $\lambda_{\mathrm{H}} := \frac{\zeta-(q-q^{-1})}{\zeta}$, we have:

Definition (Homflypt polynomial, 1984)

The 2–variable invariant $P(q,\lambda_{\mathrm{H}})$ of the oriented link L is the function:

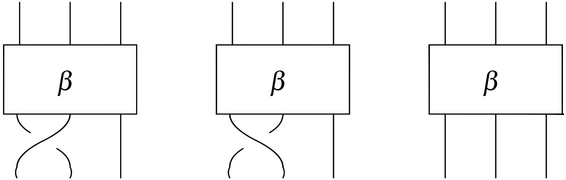
$$P(q,\lambda_{\mathrm{H}})(\hat{\alpha}) = \left(\frac{1}{\zeta\lambda_{\mathrm{H}}}\right)^{n-1} \left(\sqrt{\lambda_{\mathrm{H}}}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where $\alpha \in B_n$, $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in α and π is the natural epimorphism of $\mathbb{C}B_n$ onto $\mathrm{H}_n(q)$.

The Homflypt polynomial (skein relation)

The Homflypt polynomial satisfies the following skein relation:

$$\frac{1}{\sqrt{\lambda_{\mathrm{H}}}}P(L_+) - \sqrt{\lambda_{\mathrm{H}}}P(L_-) = (q - q^{-1})P(L_0).$$



L_+ L_- L_0

Framization of the Hecke algebra of type A

The (modular) framed braid group

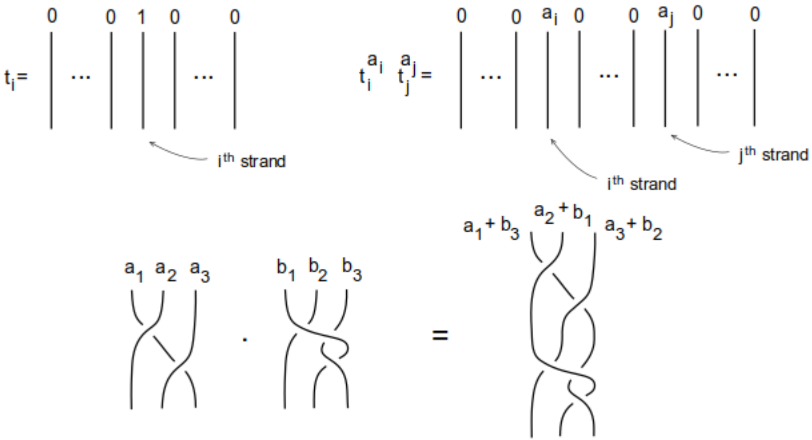
The framed braid group on n strands $\mathcal{F}_n := \mathbb{Z}^n \rtimes B_n$

Generators of $\mathcal{F}_n$ | $t_i := (1,1,\ldots,t,1,\ldots,1)$ *framing generators*
σ_i *braiding generators*

$$\mathcal{F}_n = \left\langle \sigma_1,\ldots,\sigma_{n-1},t_1,\ldots,t_n \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j|>1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j|=1 \\ t_it_j = t_jt_i \quad \sigma_it_j = t_{s_i(j)}\sigma_i & \end{array} \right\rangle$$

If we consider framings modulo d ($t_i^d = 1$) in the above presentation we obtain the *modular* framed braid group $\mathcal{F}_{d,n}$.

Diagrammatics of framed braids



The Yokonuma–Hecke algebra

$$\mathrm{Y}_{d,n}(q) = \mathrm{Alg}_{\mathbb{C}} \left\{ g_1, \ldots, g_{n-1}, t_1, \ldots, t_n \;\middle|\; \begin{array}{ll} g_i g_j = g_j g_i & |i-j| > 1 \\ g_i g_j g_i = g_j g_i g_j & |i-j| = 1 \\ t_i^d = 1, \quad t_i t_j = t_j t_i & \\ g_i t_j = t_{s_i(j)} g_i & \\ g_i^2 = 1 + (q - q^{-1}) e_i g_i & \end{array} \right\}$$

where $e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}$.

- For $d = 1$ $\mathrm{Y}_{d,n}(q)$ coincides with $\mathrm{H}_n(q)$.
- It is the basic example of framization of a knot algebra.

N. Thiem, and M. Chlouveraki & L. Poulain d'Andecy studied the Representation Theory of $\mathrm{Y}_{d,n}(u)$

Diagrammatic interpretations

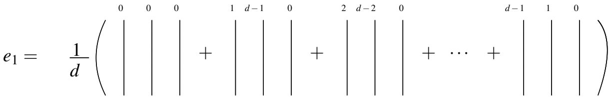


Figure: The element $e_1 \in \mathbb{C}\mathcal{F}_{d,3}$.

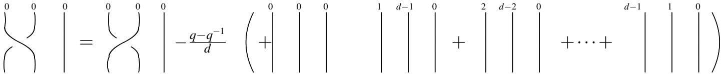


Figure: The element $g_1^{-1} \in \mathrm{Y}_{d,3}(u)$.

Markov trace on $\mathrm{Y}_{d,n}(q)$ (Juyumaya, 2004)

Let d a positive integer. For indeterminates $z, x_1, \ldots, x_{d-1}$ there exists a unique linear Markov trace tr:

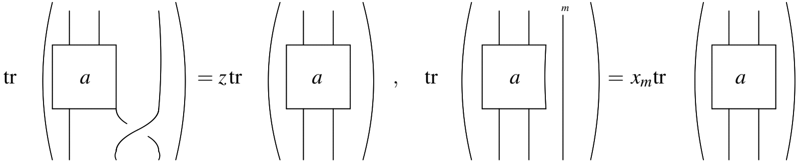
$$\mathrm{tr} : \cup_{n=1}^{\infty} \mathrm{Y}_{d,n}(q) \longrightarrow \mathbb{C}[z, x_1, \ldots, x_{d-1}]$$

defined inductively on n by the following rules:

$$\begin{array}{rcll} \mathrm{tr}(ab) & = & \mathrm{tr}(ba) & \\ \mathrm{tr}(1) & = & 1 & \\ \mathrm{tr}(a g_n) & = & z\,\mathrm{tr}(a) & \text{(Markov property)} \\ \mathrm{tr}(a t_{n+1}^s) & = & x_s \mathrm{tr}(a) & (s = 1, \ldots, d-1) \end{array}$$

where $a, b \in \mathrm{Y}_{d,n}(q)$

Topological Interpretation of the trace rules



Link invariants through tr

Isotopy classes of framed links are in bijection with equivalence classes of framed braids under:

Markov Moves

I. Conjugation: $\alpha\beta \sim \beta\alpha$, $\alpha, \beta \in \mathcal{F}_{d,n}$

II. Stabilization: $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in \mathcal{F}_{d,n}$

Markov trace

2. Conjugation: $\mathrm{tr}(ab) = \mathrm{tr}(ba)$
3. Markov Property: $\mathrm{tr}(ag_n) = z\,\mathrm{tr}(a)$

$a, b \in \mathrm{Y}_{d,n}(q)$

Link invariants from tr

Aim:

$$\mathrm{tr}(ag_n) = \mathrm{tr}(ag_n^{-1}), \qquad a \in \mathrm{Y}_{d,n}(q)$$

Problem: The trace tr does not *r*escale directly.

Answer: The parameters x_i should be solutions to the following non–linear system of equations, for any $m \in \mathbb{Z}/d\mathbb{Z}$:

$$\sum_{s=0}^{d-1} x_{m+s}x_{-s} = x_m \sum_{s=0}^{d-1} x_s x_{-s} \qquad \text{(E–system)}$$

Link invariants from tr

Solutions (Gerardín 2012): Parametrised by the non–empty subsets of $\mathbb{Z}/d\mathbb{Z}$.

$$x_D = \frac{1}{|S|} \sum_{s \in D} \exp_s$$

where: D is any non–empty subset of $\mathbb{Z}/d\mathbb{Z}$

$\exp_s(k) := \cos\frac{2\pi sk}{d} + i\sin\frac{2\pi sk}{d}$.

- If $|D| = 1$ if and only if the x_i's are d^{th} roots of unity

Let $X_D = (\mathrm{x}_1, \ldots, \mathrm{x}_{d-1})$ be a solution of the E–system parametrized by the non–empty subset D of $\mathbb{Z}/d\mathbb{Z}$.

Definition

The trace map tr_D defined as the trace tr with the parameters x_i specialized to the values x_i, shall be called the *specialized Juyumaya trace* with parameter z.

Link invariants

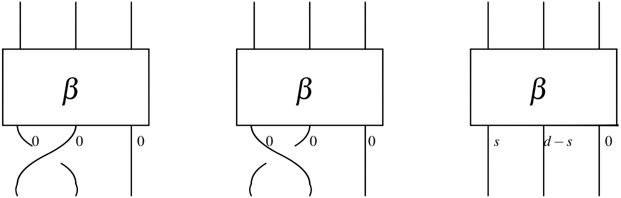
Set $E_D := \mathrm{tr}_D(e_i) = \frac{1}{|D|}$. Denoting $\lambda_D := \frac{z-(q-q^{-1})E_D}{z}$ we have for **framed links** the following (Juyumaya, Lambropoulou, 2010, 2013):

$$\Gamma_{d,D}(q,\lambda_D)(\widehat{\alpha}) = \left(\frac{1}{z\lambda_D}\right)^{n-1}\left(\sqrt{\lambda_D}\right)^{\varepsilon(\alpha)}\mathrm{tr}_D(\gamma(\alpha))$$

where γ the natural epimorphism: $\mathcal{F}_n \to \mathrm{Y}_{d,n}(u)$ and $\alpha \in \cup_\infty \mathcal{F}_n$.

A skein relation for $\Gamma_{d,D}(q,\lambda_D)$

$$\frac{1}{\sqrt{\lambda_D}}\Gamma_{d,D}(L_+) - \sqrt{\lambda_D}\Gamma_{d,D}(L_-) = \frac{q-q^{-1}}{d}\sum_{s=0}^{d-1}\Gamma_{d,D}(L_s)$$



$L_+ = \widehat{\beta\sigma_1}$ $L_- = \widehat{\beta\sigma_1{}^{-1}}$ $L_s = \widehat{\beta t_1^s t_2^{d-s}}$

Figure: The framed links L_+, L_- and L_s.

Invariants for classical links

Restrict the homomorphism γ to $\mathbb{C}B_n$ and denote it by δ. The following is an invariant of **classical links**:

$$\Delta_{d,D}(q,\lambda_D)(\widehat{\alpha}) = \left(\frac{1}{z\lambda_D}\right)^{n-1} \left(\sqrt{\lambda_D}\right)^{\varepsilon(\alpha)} \mathrm{tr}_D(\delta(\alpha))$$

- The generators t_i, the elements e_i and the skein relation of $\Gamma_{d,D}$ have no topological interpretation.
- For $d = 1$, $\Delta_{1,\{0\}}(\widehat{\alpha}) = P(\widehat{\alpha})$.

Proposition (Chlouveraki, Lambropoulou)

For $d > 1$, $\Delta_{d,D}(q,\lambda_D)$ coincides with the HOMFLYPT polynomial if and only if $|D| = 1$ or $q = 1$.

Behaviour on mirror images

1. Let $\widehat{\alpha}$ be a classical link and $\widehat{\alpha^*}$ denote its mirror image. Then:

$$\mathrm{tr}_D(q,\lambda_D)(\alpha) = \mathrm{tr}_D\left(\frac{1}{q},\frac{1}{\lambda_D}\right)(\alpha^*).$$

2. Given a solution X_D of the E–system, for any braid $\alpha \in B_n$, we have that:

$$\Delta_{d,D}(\widehat{\alpha^*})(q,\lambda_D) = \Delta_{d,D}(\widehat{\alpha})\left(\frac{1}{q},\frac{1}{\lambda_D}\right).$$

Recent Developments

On **classical links** the trace tr_D can be computed only by using rules involving the generators g_i and the elements e_i.

Theorem* (Juyumaya, Karvounis, Lambropoulou)

Let X_D be a solution of the E*–system and let $b \in B_{n+1}$. Then the specialized Juyumaya trace $\mathrm{tr}_D(\delta(b))$ can be computed by using only the following rules:*

$$\begin{array}{rcll}
\mathrm{tr}_D(\alpha\beta) & = & \mathrm{tr}_D(\beta\alpha) & (\alpha,\beta \in \mathrm{Y}_{d,n}(q)) \\
\mathrm{tr}_D(1) & = & 1 & (1 \in \mathrm{Y}_{d,n}(q)) \\
\mathrm{tr}_D(\alpha g_n) & = & z\,\mathrm{tr}_D(\alpha) & (\alpha \in \mathrm{Y}_{d,n}(q)) \ \textit{(Markov property)} \\
\mathrm{tr}_D(\alpha e_n) & = & E_D\,\mathrm{tr}_D(\alpha) & (\alpha \in \mathrm{Y}_{d,n}(q)) \\
\mathrm{tr}_D(\alpha e_n g_n) & = & z\,\mathrm{tr}_D(\alpha) & (\alpha \in \mathrm{Y}_{d,n}(q)).
\end{array}$$

*Conjectured by J. Juyumaya.

Behaviour on knots

Theorem* (Chlouveraki, Jablan, Karvounis, Lambropoulou)

Given a solution X_D of the E*–system, for any braid $\alpha \in B_n$ such that $\widehat{\alpha}$ is a **knot**, we have that:*

$$\Delta_d(\widehat{\alpha})(q,z) = P(\widehat{\alpha})(q, \tfrac{z}{E_D}),$$

or equivalently: $\Delta_d(\widehat{\alpha})(q,\lambda_D) = P(\widehat{\alpha})(q,\lambda_D)$.

This is because $\mathrm{tr}_D(\alpha) = E_D^{n-1}\tau(\alpha)$, where n the number of components of $\widehat{a}$ and τ the Ocneanu trace at variable $\frac{z}{E_D}$.

*Conjectured by S. Jablan and K. Karvounis.

Behaviour on disjoint links

Theorem (Karvounis)

Given a solution X_D of the E*–system, for any braid $\alpha \in B_n$ such that $\widehat{\alpha}$ is a* ***disjoint union of*** k ***knots****, we have that:*

$$\Delta_d(\widehat{\alpha})(q,z) = E_D^{1-k}\Delta_1(\widehat{\alpha})(q,\tfrac{z}{E_D}) = E_D^{1-k}P(\widehat{\alpha})(q,\tfrac{z}{E_D}).$$

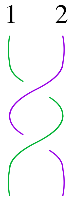


Figure: The braid σ_1^2

Example: the Hopf link $\widehat{\sigma_1^2}$.

$$\mathrm{tr}_D(g_1^2) = E_D \tau(g_1^2) + 1 - E_D$$

hence:

$$\Delta_d(\widehat{\sigma_1^2}) = P(\widehat{\sigma_1^2}) + E_D^{-1} \lambda_D (1 - E_D) P(\widehat{1_2}),$$

where 1_2 is the identity braid on 2 strands.

Behaviour on links

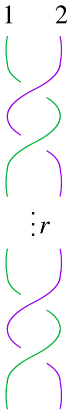


Figure: The braid σ_m^{2k}

Example:

$$\mathrm{tr}_D(g_m^{2k}) = E_D \tau(g_m^{2k}) + 1 - E_D,$$

hence:

$$\Delta_d(\widehat{\sigma_m^{2k}}) = E_D^{1-n}\left[E_D P(\widehat{\sigma_m^{2k}}) + \lambda_D^k (1 - E_D) P(\widehat{1_n})\right],$$

where 1_n is the identity braid on n strands.

- There is an extra term, corresponding to the Homflypt polynomial of the unlink.
- More complicated on other examples.

Behaviour of tr_D on the elements e_i

The elements e_i have different behaviour depending on whether the i-th and $(i+1)$-st strands belong to the same component.

For **knots**, we know that:

$$\mathrm{tr}_D(\alpha e_i) = \mathrm{tr}_D(\alpha).$$

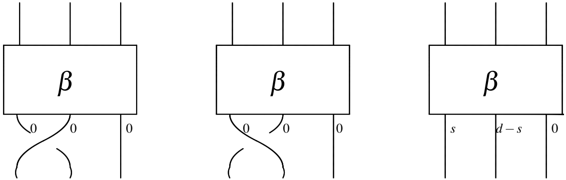
On **links**, it does not seem to hold. Example:

$$\mathrm{tr}_D(e_1 g_1^{2k}) = \mathrm{tr}_D(g_1^{2k}) + E_D - 1.$$

A special skein relation

The skein relation of $\Gamma_{d,D}$ involves the following links:

$$\frac{1}{\sqrt{\lambda_D}}\Gamma_{d,D}(L_+) - \sqrt{\lambda_D}\Gamma_{d,D}(L_-) = \frac{q-q^{-1}}{d}\sum_{s=0}^{d-1}\Gamma_{d,D}(L_s)$$



$L_+ = \widehat{\beta\sigma_1}$ $\quad L_- = \widehat{\beta\sigma_1{}^{-1}}$ $\quad L_s = \widehat{\beta t_1^s t_2^{d-s}}$

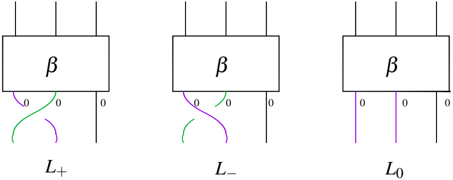
- If the crossing of L_+ (and so of L_-) involves two different components, then the strands on L_s belong to the same component.
- Hence, the framings add themselves and, modulo d, the link L_s reduces to L_0, which is a classical link.

A special skein relation (2)

Proposition (Karvounis)

If the skein relation of $\Gamma_{d,D}$ is applied on a link crossing, then it reduces to the skein relation of the Homflypt polynomial:

$$\frac{1}{\sqrt{\lambda_D}}\Gamma_{d,D}(L_+) - \sqrt{\lambda_D}\Gamma_{d,D}(L_-) = (q - q^{-1})\Gamma_{d,D}(L_0).$$



Since the skein relation involves only classical links, it holds also for Δ_d.

Comparison with the Homflypt on links (3)

Theorem (Karvounis, Lambropoulou)

Let $\mathcal{A} = \mathbb{Q}[q^{\pm 1}, \sqrt{\lambda_D}^{\pm 1}]$. For any ℓ-component link L, $\Delta_d(L)$ is an $\mathcal{A}$-linear combination of $P(L)$ and the values of P on disjoint unions of knots obtained by the skein relation:

$$\Delta_d(L) = P(L) + \sum_{k=2}^{\ell} (E_D^{1-k} - 1) \sum_{\widehat{\alpha} \in \mathcal{N}(L)_k} c(\widehat{\alpha})\, P(\widehat{\alpha}),$$

where $\mathcal{N}(L)_k$ is the set of all disjoint unions of k knots appearing in this linear combination.

Distinguishing P-equivalent links

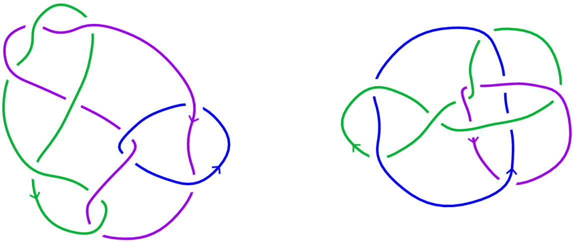
- Up to 11 crossings, there are 89 pairs of P-equivalent links which are not isotopic as unoriented links (data from LinkInfo [Cha, Livingston]).
- To check if there are P-equivalent but non Δ_d-equivalent links, we developed computer programs.
- Out of these 89, the following 6 pairs of (3-component) links are *not* Δ_d-equivalent for every $d \geq 2$:

$L11n358\{0,1\}$	$L11n418\{0,0\}$
$L11a467\{0,1\}$	$L11a527\{0,0\}$
$L11n325\{1,1\}$	$L11n424\{0,0\}$
$L10n79\{1,1\}$	$L10n95\{1,0\}$
$L11a404\{1,1\}$	$L11a428\{0,1\}$
$L10n76\{1,1\}$	$L11n425\{1,0\}$

It can be proved diagrammatically that the six pairs are not Δ_d-equivalent.

Distinguishing P-equivalent links (2)

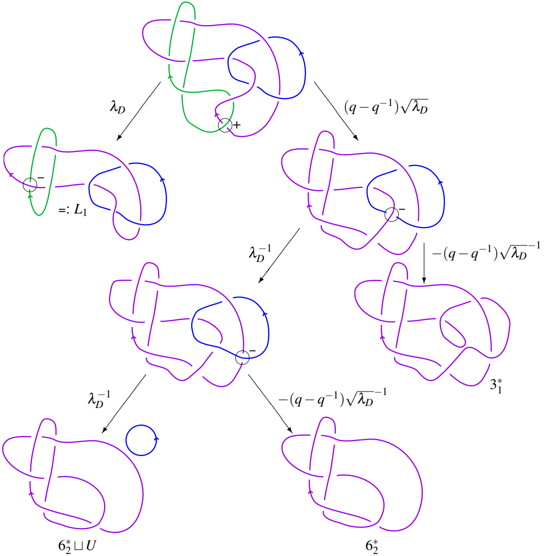
A diagrammatic proof has been completed for the first pair of links ($L11n358\{0,1\}$ and $L11n418\{0,0\}$).



Theorem (Karvounis, Lambropoulou)

The invariants Δ_d, $d \geq 2$, are not topologically equivalent to P.

Example of the computation for the link $L11n358\{0,1\}$



Framization of the Temperley–Lieb algebra

The Temperley–Lieb algebra

$$\mathrm{TL}_n(q) = \mathrm{Alg}_{\mathbb{C}} \left\{ h_1, \ldots, h_{n-1} \middle| \begin{array}{cc} h_i h_j = h_j h_i & |i-j| > 1 \\ h_i h_j h_i = h_i h_j h_i & |i-j| = 1 \\ h_i^2 = 1 + (q - q^{-1}) h_i & \\ h_{i,j} = 0 & |i-j| = 1 \end{array} \right\}$$

where

$$h_{i,j} := 1 + q(h_i + h_j) + q^2(h_i h_j + h_j h_i) + q^3(h_i h_j h_i)$$

The Temperley–Lieb algebra

The Temperley–Lieb algebra as a quotient of $\mathrm{H}_n(q)$:

$$\mathrm{TL}_n(q) = \frac{\mathrm{H}_n(q)}{\langle h_{i,j} \,\big|\, |i-j| = 1 \rangle}$$

The ideal $\langle h_{i,j} \,\big|\, |i-j| = 1 \rangle$ is principal, generated by $h_{1,2}$. That is:

$$h_{1,2} = 1 + q(h_1 + h_2) + q^2(h_1h_2 + h_2h_1) + q^3 h_1h_2h_1$$

Jones' trace on TL_n

Markov trace on $\mathrm{TL}_n(q)$

The trace τ passes to the quotient algebra $\mathrm{TL}_n(q)$ if and only if:

$$\tau(h_{1,2}) = 0$$

or equivalently:

$$(q^2+1)q^2\zeta^2 + (q^2+2)q\zeta + 1 = 0$$

$$\zeta = -\frac{1}{q(q^2+1)} \quad \text{or} \quad \zeta = -\frac{1}{q}$$

The Jones polynomial

Specializing $\zeta = -\frac{1}{q(q^2+1)}$ in $P(q,\lambda_H)$ we obtain the **Jones polynomial** (1984):

$$V(q)(\hat{\alpha}) = \left(-\frac{1}{\lambda_H q^2}\right)^{n-1} q^{2\varepsilon(\alpha)} \tau(\pi(\alpha)) = P(q,q^4)(\hat{\alpha})$$

Classical case	$\mathrm{H}_n(q)$	quotient of $\mathbb{C}B_n$.
	$\mathrm{TL}_n(q)$	def. ideal generated by $h_{1,2}$

Framed case	$\mathrm{Y}_{d,n}(q)$	quotient of $\mathbb{C}\mathcal{F}_{d,n} := \mathbb{C}\left[C_d^n \rtimes B_n\right]$.
	$\mathrm{FTL}_{d,n}(q)$	def. ideal generated by $e_1 e_2 g_{1,2}$.

Proposition (Chlouveraki, Pouchin 2013)

The dimension of $\mathrm{FTL}_{d,n}(u)$ *is:*

$$\dim \mathrm{FTL}_{d,n}(u) = \sum_{|k_1|+|k_2|+\ldots+|k_d|=n} \left(\frac{n!}{k_1!\ldots k_d!} \right)^2 c_{k_1} \ldots c_{k_d}$$

where c_k *is the* k^{th} *Catalan number.*

Proposition (DG)

The following set is a linear basis for $\mathrm{FTL}_{2,3}(u)$*:*

$$\{ 1, t_1, t_2, t_3, t_1t_2, , t_1t_3, t_2t_3, t_1t_2t_3,$$
$$g_1, t_1g_1, t_2g_1, t_3g_1, t_1t_2g_1, t_1t_3g_1, t_2t_3g_1, t_1t_2t_3g_1,$$
$$g_2, t_1g_2, t_2g_2t_3g_2, t_1t_2g_2, t_1t_3g_2, t_2t_3g_2t_1t_2t_3g_2,$$
$$g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,$$
$$g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1$$
$$t_1g_1g_2g_2, t_2g_1g_2g_1, t_3g_1g_2g_1, t_1t_2g_1g_2g_1, t_1t_3g_1g_2g_1, t_2t_3g_1g_2g_1 \}.$$

A presentation with non-invertible generators

Proposition (DG)

The algebra $\mathrm{FTL}_{d,n}(u)$ *can be presented with generators:*

$$1, \ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$t_i^d = 1, \quad t_i t_j = t_j t_i$$

$$\ell_i t_j = t_j \ell_i, \quad \ell_i t_j = t_j \ell_i, \quad |i-j| > 1$$

$$\ell_i \ell_j = \ell_j \ell_i, \quad |i-j| > 1$$

$$\ell_i t_i = t_{i+1} \ell_i + \frac{1}{q^2+1}(t_i - t_{i+1})$$

$$\ell_i t_{i+1} = t_i \ell_i + \frac{1}{q^2+1}(t_{i+1} - t_i)$$

$$\ell_i^2 = \frac{(q^2-1)e_i + 2}{u+1}\,\ell_i,$$

$$\ell_i \ell_{i+1} \ell_i - \frac{(q^2-1)+1}{(q^2+1)^2}\,\ell_i = \ell_{i+1} \ell_i \ell_{i+1} - e_{i+1} e_i \frac{(q^2-1)+1}{(q^2+1)^2}\,\ell_{i+1}$$

$$e_i e_{i+1} \ell_i \ell_{i+1} \ell_i = e_i e_{i+1} \frac{(q^2-1)+1}{(q^2+1)^2}\,\ell_i$$

The case of $\mathrm{FTL}_{d,n}(u)$

Theorem F (DG, Juyumaya, Kontogeorgis, Lambropoulou, 2014)

The trace tr *passes to the quotient algebra* $\mathrm{FTL}_{d,n}(u)$ *if and only if the trace parameters satisfy:*

$$x_m = -z\left(\sum_{s\in D_1} \exp_s(m) + \sum_{s\in D_2} \exp_s(m)\right) \quad \text{and} \quad z = -\frac{1}{q(|D_1|+(q^2+1)|D_2|)}$$

where the disjoint union $D_1 \cup D_2$ *is the support of the Fourier transform* $\hat{x}$ *of* x*,*

$$D_1 = \{s \in \mathbb{Z}/d\mathbb{Z} \,|\, y_s = -dz\}, \quad D_2 = \{s \in \mathbb{Z}/d\mathbb{Z} \,|\, y_s = -dz(q^2+1)\}$$

Sketch of proof

- Let tr kill the generator of the defining ideal and obtain conditions for z.
- Consider x_m as the value at m of the complex function x over $\mathbb{Z}/d\mathbb{Z}$.
- Use tools of Harmonic Analysis of Finite Groups.

Corollary (DG, Kontogeorgis)

In the case where one of the sets S_1 or S_2 is the empty set we obtain that the values in Theorem F become solutions of the E–system.

	x_m	z
$D_1 = \emptyset$	$\sum_{s \in D_2} \exp_s(m)$	$-\frac{1}{q(q^2+1)\lvert D_2 \rvert}$
$D_2 = \emptyset$	$\sum_{s \in D_1} \exp_s(m)$	$-\frac{1}{q\lvert D_1 \rvert}$

Link Invariants from $\mathrm{FTL}_{d,n}(q)$

For framed links:

$$\vartheta_{d,D}(q)(\widehat{\alpha}) = \left(-(q+q^{-1})|D|\right)^{n-1} q^{2\varepsilon(\alpha)}\mathrm{tr}_D(\alpha) = \Gamma_{d,D}(q,q^4)(\widehat{\alpha}),$$

and for classical links:

$$\theta_d(q)(\widehat{\alpha}) = \left(-(q+q^{-1})|D|\right)^{n-1} q^{2\varepsilon(\alpha)}\mathrm{tr}_D(\alpha) = \Delta_d(q,q^4)(\widehat{\alpha}).$$

The invariants θ_d distinguish the same six pairs as the invariants Δ_d:

$L11n358\{0,1\}$	$L11n418\{0,0\}$
$L11a467\{0,1\}$	$L11a527\{0,0\}$
$L11n325\{1,1\}$	$L11n424\{0,0\}$
$L10n79\{1,1\}$	$L10n95\{1,0\}$
$L11a404\{1,1\}$	$L11a428\{0,1\}$
$L10n76\{1,1\}$	$L11n425\{1,0\}$

Further Work

- Investigation of the 3-manifold invariants related to $\mathrm{FTL}_{d,n}$.
- Statistical mechanics model from the invariants θ_d.
- Transverse link invariants.

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Thank you very much for your attention!