Maximal operator on variable exponent spaces

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Lebesgue spaces $L^{p(.)}(\Omega)$

Definition

Let $\Omega \subset \mathbb{R}^n$ and $p(.) : \Omega \to [1, \infty)$ be a measurable function. Define for a measurable function $f : \Omega \to \mathbb{R}$

a convex modular

$$m_{L^{p(.)}(\Omega)} := \int_\Omega |f(x)|^{p(x)} dx,$$

a norm by

$$\|f\|_{L^{p(.)}(\Omega)} := \inf\left\{\lambda > 0; \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

and a space $L^{p(.)}(\Omega)$

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Define for $f \in L^1_{loc}(\mathbb{R}^n)$ *the maximal operator by*

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x)| dx.$$

Theorem (Classical theorem)

Let $1 < p \leq \infty$ *be a number. Then*

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Let $p(.) : \mathbb{R}^n \to \mathbb{R}$, $1 < p_* \le p(x) \le p^* < \infty$. Say that $p(.)$ is log-*Hölder continuous, write $p(.) \in \log -\mathcal{H}$, if there exists a $K > 0$ such that*

$$|x - y| \le \frac{1}{e} \implies |p(x) - p(y)| \le \frac{K}{\ln \frac{1}{|x-y|}}$$

Theorem (Diening - 2004)

Let $p(.)$ is log-*Hölder continuous and $p(.)$ is constant function near infinity, i.e. there is $R >$ and $1 < p_\infty < \infty$ such that $p(x) = p_\infty$ for $|x| > R$. Then*

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Definition

Say that $p(.) \in \mathcal{N}$ *if there exist constants* $0 < c < 1$ *and* $1 < p_\infty$ *such that*

$$\int_{\mathbb{R}} c^{\frac{1}{|p(x)-p_\infty|}}\, dx < \infty,$$

where we use a convention $c^{1/0} = 0$*.*

Theorem (Nekvinda - 2004)

Let $p(.) \in \log -\mathcal{H} \cap \mathcal{N}$*. Then*

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Theorem (D. Cruiz-Uribe, A. Fiorenza, C.J. Neugebauer - 2003, 2004)

Let $p(.) \in \log -\mathcal{H}$ *and let* $p(.) \in \mathcal{UFN}$*, i.e. there exist numbers* $K > 0, L > 0, p_\infty > 1$ *such that*

$$|p(x) - p_\infty| \leq \frac{K}{\ln |x|}, \quad |x| \geq L,$$

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Let $s(.) : [0, \infty) \to \mathbb{R}$, $1 < s_* \le s(t) \le s^* < \infty$. *Say that* $s(.) \in \mathcal{B}$ *if* $s(.)$ *is monotone and there exist* $L > 1$ *and* $\alpha > 0$ *such that*

$$\left|\frac{ds}{dt}(t)\right| \le \frac{L}{t|\ln t|^{1+\alpha}}.$$

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Say that $p(.) \in \mathcal{B}$, *if* $p(x) = s(|x|)$ *for some* $s(.) \in \mathcal{B}$.

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Example

Let $\alpha > 0, L > 0$ *and* $p(.) = p_\infty + \frac{L}{(\ln|x|)^\alpha}$ *or* $p(.) = p_\infty - \frac{L}{(\ln|x|)^\alpha}$ *for large* $|x|$. *Then*

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We know this the previous example for $\alpha \geq 1$. *This theorem is new for* $0 < \alpha < 1$.

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Let $\alpha > 0, L > 0$ *and* $p(.) = p_\infty + \frac{\sin(20 \ln \ln |x|)}{(\ln |x|)^\alpha}$ *for large* $|x|$. *Then*

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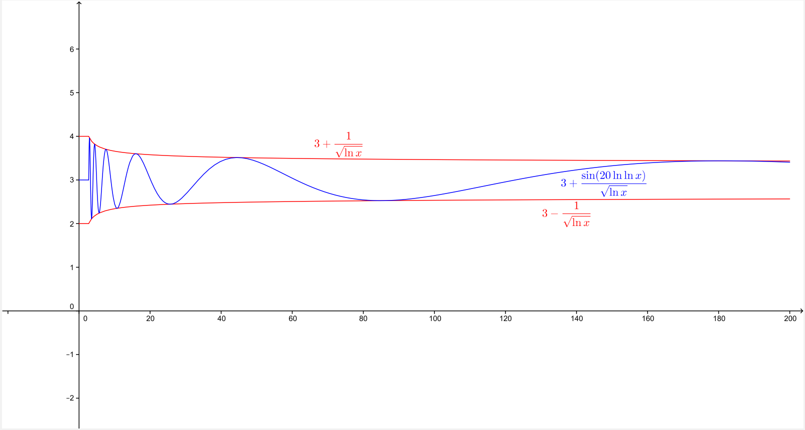
Examples of functions



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Thank you for your attention