Modular eigenvalues of the variable exponent p-Laplacian

Joint Meeting AMS-EMS-SPMM
Porto, June 2015

Osvaldo Méndez

University of Texas-El Paso

Table of contents

D. Edmunds, J. Lang, O.M.

,

Differential Operators on Spaces of Variable Integrability

D. Edmunds, J. Lang, O.M.

,

Differential Operators on Spaces of Variable Integrability

J. Lang, O. Méndez

D. Edmunds, J. Lang, O.M.

,

Differential Operators on Spaces of Variable Integrability

J. Lang, O. Méndez

Nonlinear An., 2014

,

Journal of Diff. Eq, 2015

,

Journal d'Analyse Mathematique, 2015

Preliminaries

$\Omega \subset \mathbb{R}^n$ bounded, Lipschitz domain,

$$p : \Omega \longrightarrow (1, \infty)$$

Borel-measurable.

Preliminaries

$\Omega \subset \mathbb{R}^n$ bounded, Lipschitz domain,

$$p : \Omega \longrightarrow (1, \infty)$$

Borel-measurable.

$$\rho_{p(\cdot),\Omega}(f) = \int_{\Omega} |f(x)|^{p(x)}\, dx$$

is a left-continuous convex modular on $\mathcal{M}(\Omega)$; $\mathcal{M}(\Omega)$ being the set of all Lebesgue-measurable, scalar-valued functions on Ω, $f = g$ iff $f = g$ a.e..

Preliminaries

$\Omega \subset \mathbb{R}^n$ bounded, Lipschitz domain,

$$p : \Omega \longrightarrow (1, \infty)$$

Borel-measurable.

$$\rho_{p(\cdot),\Omega}(f) = \int_\Omega |f(x)|^{p(x)}\, dx$$

is a left-continuous convex modular on $\mathcal{M}(\Omega)$; $\mathcal{M}(\Omega)$ being the set of all Lebesgue-measurable, scalar-valued functions on Ω, $f = g$ iff $f = g$ a.e..

$$\|f\|_{p(\cdot),\Omega} := \inf \left\{\lambda > 0 : \rho_{p(\cdot),\Omega}(f/\lambda) \leq 1\right\}$$

is a norm on the space

$$L^{p(\cdot)}(\Omega) = \left\{f \in \mathcal{M}(\Omega) : \rho_{p(\cdot),\Omega}(\lambda f) < \infty \text{ for some } \lambda > 0\right\}.$$

Particular case of Musielak-Orlicz spaces.

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

$$W_0^{1,p(\cdot)}(\Omega)$$

is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

$$W_0^{1,p(\cdot)}(\Omega)$$

is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. If $p \in C(\overline{\Omega})$ the embedding

$$E : W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

$$W_0^{1,p(\cdot)}(\Omega)$$

is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. If $p \in C(\overline{\Omega})$ the embedding

$$E : W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact and Poincarés inequality holds:

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

$$W_0^{1,p(\cdot)}(\Omega)$$

is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. If $p \in C(\overline{\Omega})$ the embedding

$$E : W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact and Poincarés inequality holds:

$$\|u\|_{p(\cdot)} \leq C(p, n, \Omega)\||\nabla u|\|_{p(\cdot)}.$$

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

$$W_0^{1,p(\cdot)}(\Omega)$$

is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. If $p \in C(\overline{\Omega})$ the embedding

$$E : W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact and Poincarés inequality holds:

$$\|u\|_{p(\cdot)} \leq C(p, n, \Omega)\||\nabla u|\|_{p(\cdot)}.$$

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

$$W_0^{1,p(\cdot)}(\Omega)$$

is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. If $p \in C(\overline{\Omega})$ the embedding

$$E : W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact and Poincarés inequality holds:

$$\|u\|_{p(\cdot)} \leq C(p, n, \Omega)\||\nabla u|\|_{p(\cdot)}.$$

$$\|u\| = \||\nabla u|\|_{p(\cdot)}$$

Sobolev Spaces, Poincarés inequality, Sobolev Embedding.

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot),\Omega} + \||\nabla u|\|_{p(\cdot),\Omega}.$$

$$W_0^{1,p(\cdot)}(\Omega)$$

is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. If $p \in C(\overline{\Omega})$ the embedding

$$E : W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact and Poincarés inequality holds:

$$\|u\|_{p(\cdot)} \leq C(p, n, \Omega)\||\nabla u|\|_{p(\cdot)}.$$

$$\|u\| = \||\nabla u|\|_{p(\cdot)}$$

is a norm on $W_0^{1,p(\cdot)}(\Omega)$ equivalent to the original one.

The THERMISTOR PROBLEM (Zhikov, 1996)

$$\Delta_{p(\cdot)} u := \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = 0$$

The THERMISTOR PROBLEM (Zhikov, 1996)

$$\Delta_{p(\cdot)} u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = 0$$

Existence, uniqueness and regularity of the solution!!!

The THERMISTOR PROBLEM (Zhikov, 1996)

$$\Delta_{p(\cdot)} u := \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = 0$$

Existence, uniqueness and regularity of the solution!!!

Fan-Zhang, 2003

The THERMISTOR PROBLEM (Zhikov, 1996)

$$\Delta_{p(\cdot)} u := \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = 0$$

Existence, uniqueness and regularity of the solution!!!
Fan-Zhang, 2003
Modeling of Non-Newtonian fluid hydrodynamics,
Electrorheological fluids.

The THERMISTOR PROBLEM (Zhikov, 1996)

$$\Delta_{p(\cdot)} u := \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u \right) = 0$$

Existence, uniqueness and regularity of the solution!!!

Fan-Zhang, 2003

Modeling of Non-Newtonian fluid hydrodynamics,

Electrorheological fluids.

Electrorheological fluids: Modeling and Mathematical Theory.

Rŭčiǩa

Modular Eigenvalue problem

$$-\Delta_{p(\cdot)}u = \lambda|u|^{p(\cdot)-2}u \quad (1)$$

Modular Eigenvalue problem

$$-\Delta_{p(\cdot)} u = \lambda |u|^{p(\cdot)-2} u \qquad (1)$$

(Fan-Zhang-Zhao, 2005)

Modular Eigenvalue problem

$$-\Delta_{p(\cdot)} u = \lambda |u|^{p(\cdot)-2} u \tag{1}$$

(Fan-Zhang-Zhao, 2005)

$$\int_\Omega |\nabla u|^{p(\cdot)-2} \nabla u \nabla h = \lambda \int_\Omega |u|^{p(\cdot)-2} uh \tag{2}$$

$\forall h \in C_0^\infty(\Omega)$.

Modular Eigenvalue problem

$$-\Delta_{p(\cdot)}u = \lambda |u|^{p(\cdot)-2}u \tag{1}$$

(Fan-Zhang-Zhao, 2005)

$$\int_\Omega |\nabla u|^{p(\cdot)-2}\nabla u \nabla h = \lambda \int_\Omega |u|^{p(\cdot)-2}uh \tag{2}$$

$\forall h \in C_0^\infty(\Omega)$.

If p is constant $\to$ first positive eigenvalue:

$$\lambda = \frac{1}{\|E\|^p}.$$

Modular Eigenvalue problem

$$-\Delta_{p(\cdot)} u = \lambda |u|^{p(\cdot)-2} u \tag{1}$$

(Fan-Zhang-Zhao, 2005)

$$\int_\Omega |\nabla u|^{p(\cdot)-2} \nabla u \nabla h = \lambda \int_\Omega |u|^{p(\cdot)-2} uh \tag{2}$$

$\forall h \in C_0^\infty(\Omega)$.

If p is constant $\rightarrow$ first positive eigenvalue:

$$\lambda = \frac{1}{\|E\|^p}.$$

If p has either a local min or a local max in Ω.........

Modular Eigenvalue problem

$$-\Delta_{p(\cdot)}u = \lambda|u|^{p(\cdot)-2}u \qquad (1)$$

(Fan-Zhang-Zhao, 2005)

$$\int_\Omega |\nabla u|^{p(\cdot)-2}\nabla u \nabla h = \lambda \int_\Omega |u|^{p(\cdot)-2}uh \qquad (2)$$

$\forall h \in C_0^\infty(\Omega)$.
If p is constant $\to$ first positive eigenvalue:

$$\lambda = \frac{1}{\|E\|^p}.$$

If p has either a local min or a local max in Ω.........

$$\inf_{0\neq u\in W_0^{1,p(\cdot)}(\Omega)} \frac{\int_\Omega |\nabla u(x)|^{p(x)}}{\int_\Omega |u(x)|^{p(x)}} = 0!!!!.$$

(Fhang-Zhang-Zhao (2005).)

The modular eigenvalue problem as Euler-Lagrange Equations

Does the modular eigenvalue problem have a solution (λ, u)?

The modular eigenvalue problem as Euler-Lagrange Equations

Does the modular eigenvalue problem have a solution (λ, u)?
Abstract theory of Ljusternik-Schnirelman:

The modular eigenvalue problem as Euler-Lagrange Equations

Does the modular eigenvalue problem have a solution (λ, u)?
Abstract theory of Ljusternik-Schnirelman: $k = 1, 2, 3...$

$$\lambda_k = \inf_{C \in \mathcal{C}_k} \sup_{v \in C} \frac{\int_\Omega |\nabla u|^{p(\cdot)}}{\int_\Omega |u|^{p(\cdot)}}.$$

The modular eigenvalue problem as Euler-Lagrange Equations

Does the modular eigenvalue problem have a solution (λ, u)?
Abstract theory of Ljusternik-Schnirelman: $k = 1, 2, 3...$

$$\lambda_k = \inf_{C \in \mathcal{C}_k} \sup_{v \in C} \frac{\int_\Omega |\nabla u|^{p(\cdot)}}{\int_\Omega |u|^{p(\cdot)}}.$$

$$\mathcal{C}_k =: \{C \subset W_0^{1,p(\cdot)}(\Omega)\,,\ C \text{ closed}\,,\ \gamma(C) \geq k\}.$$

The modular eigenvalue problem as Euler-Lagrange Equations

Does the modular eigenvalue problem have a solution (λ, u)?
Abstract theory of Ljusternik-Schnirelman: $k = 1, 2, 3...$

$$\lambda_k = \inf_{C \in \mathcal{C}_k} \sup_{v \in C} \frac{\int_\Omega |\nabla u|^{p(\cdot)}}{\int_\Omega |u|^{p(\cdot)}}.$$

$$\mathcal{C}_k =: \{C \subset W_0^{1,p(\cdot)}(\Omega)\,,\ C \text{ closed}\,,\ \gamma(C) \geq k\}.$$

γ : Krasnoselskii genus.

The modular eigenvalue problem as Euler-Lagrange Equations

Does the modular eigenvalue problem have a solution (λ, u)?
Abstract theory of Ljusternik-Schnirelman: $k = 1, 2, 3...$

$$\lambda_k = \inf_{C \in \mathcal{C}_k} \sup_{v \in C} \frac{\int_\Omega |\nabla u|^{p(\cdot)}}{\int_\Omega |u|^{p(\cdot)}}.$$

$$\mathcal{C}_k =: \{C \subset W_0^{1,p(\cdot)}(\Omega)\,,\ C \text{ closed}\,,\ \gamma(C) \geq k\}.$$

γ : Krasnoselskii genus.

Not a tangible description of the eigenfunctions.

The modular eigenvalue problem as Euler-Lagrange Equations

Does the modular eigenvalue problem have a solution (λ, u)?
Abstract theory of Ljusternik-Schnirelman: $k = 1, 2, 3...$

$$\lambda_k = \inf_{C \in \mathcal{C}_k} \sup_{v \in C} \frac{\int_\Omega |\nabla u|^{p(\cdot)}}{\int_\Omega |u|^{p(\cdot)}}.$$

$$\mathcal{C}_k =: \{C \subset W_0^{1,p(\cdot)}(\Omega)\,,\; C \text{ closed}\,,\; \gamma(C) \geq k\}.$$

γ : Krasnoselskii genus.

Not a tangible description of the eigenfunctions.
There are eigenvalues "per se" that are NOT obtained via Ljiusternik-Schnirelmann. (Bining-Rynne(2008))

Opt for an ad-hoc treatment (Sobolev embedding)

$$F : W_0^{1,p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$$
$$F(u) = \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}\, dx$$

Opt for an ad-hoc treatment (Sobolev embedding)

$$F : W_0^{1,p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$$
$$F(u) = \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}\, dx$$

$$G : L^{p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$$
$$G(u) = \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)}\, dx$$

Then

$$\langle F'(u), h \rangle = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla h(x)\, dx \tag{3}$$

and

$$\langle G'(u), h \rangle = \int_\Omega |u(x)|^{p(x)-2} u(x) h(x)\, dx \tag{4}$$

The modular eigenvalue problem is the Euler-Lagrange equation for the constrained problem

$$\max G(u) \ \text{subject to} \ F(u) = constant \tag{5}$$

Existence of the first eigenvalue

Existence of the first eigenvalue

$r > 0 \Rightarrow \exists (u_0, \lambda_0)$ solution of modular eigenvalue problem.

Existence of the first eigenvalue

$r > 0 \Rightarrow \exists (u_0, \lambda_0)$ solution of modular eigenvalue problem.

$$\int_\Omega \frac{|u_0(x)|^{p(x)}}{p(x)}\,dx = \sup_{\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}\,dx \le r} \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)}\,dx.$$

Existence of the first eigenvalue

$r > 0 \Rightarrow \exists (u_0, \lambda_0)$ solution of modular eigenvalue problem.

$$\int_\Omega \frac{|u_0(x)|^{p(x)}}{p(x)}\,dx = \sup_{\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}\,dx \le r} \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)}\,dx.$$

$$\int_\Omega \frac{|\nabla u_0(x)|^{p(x)}}{p(x)}\,dx = r$$

Stability

$$(\lambda_p, u_p)$$

$$(\lambda_q, u_q)$$

eigenvalues corresponding to p, q;

Stability

$$(\lambda_p, u_p)$$

$$(\lambda_q, u_q)$$

eigenvalues corresponding to p, q;

$$p \approx q$$

$\Rightarrow$

Stability

$$(\lambda_p, u_p)$$

$$(\lambda_q, u_q)$$

eigenvalues corresponding to p, q;

$$p \approx q$$

$\Rightarrow$

$$\lambda_p \approx \lambda_q??$$

Stability

$$(\lambda_p, u_p)$$

$$(\lambda_q, u_q)$$

eigenvalues corresponding to p, q;

$$p \approx q$$

$\Rightarrow$

$$\lambda_p \approx \lambda_q??$$

$$u_p \approx u_q??$$

(Lindqvist (1990), p constant.)

Hölder's inequality:

Hölder's inequality:

$$\int_{\Omega} |f(x)g(x)|\, dx \leq (1 + 1/p_- - 1/p_+) \|f\|_p \|g\|_{p'} .$$

Hölder's inequality:

$$\int_{\Omega} |f(x)g(x)|\, dx \leq (1 + 1/p_{-} - 1/p_{+}) \|f\|_{p} \|g\|_{p'} \, .$$

Not suitable for "stability" analysis.

Hölder's inequality:

$$\int_{\Omega} |f(x)g(x)|\, dx \leq (1 + 1/p_- - 1/p_+) \|f\|_p \|g\|_{p'} \,.$$

Not suitable for "stability" analysis.

However: If

$$p(x) \leq q(x) \leq p(x) + \varepsilon \text{ a.e. in } \Omega. \tag{6}$$

Hölder's inequality:

$$\int_\Omega |f(x)g(x)|\, dx \le (1 + 1/p_- - 1/p_+)\, \|f\|_p\, \|g\|_{p'}\,.$$

Not suitable for "stability" analysis.

However: If

$$p(x) \le q(x) \le p(x) + \varepsilon \text{ a.e. in } \Omega. \tag{6}$$

$\rho_q(f) \le 1$, then

$$\rho_p(f) \le \varepsilon\, |\Omega| + \varepsilon^{-\varepsilon}.$$

Hölder's inequality:

$$\int_\Omega |f(x)g(x)|\,dx \le (1 + 1/p_- - 1/p_+)\,\|f\|_p\,\|g\|_{p'}\,.$$

Not suitable for "stability" analysis.

However: If

$$p(x) \le q(x) \le p(x) + \varepsilon \text{ a.e. in } \Omega. \tag{6}$$

$\rho_q(f) \le 1$, then

$$\rho_p(f) \le \varepsilon\,|\Omega| + \varepsilon^{-\varepsilon}.$$

Edmunds-Lang-Nekvinda, 2009

Hölder's inequality:

$$\int_\Omega |f(x)g(x)|\, dx \le (1 + 1/p_- - 1/p_+) \|f\|_p \|g\|_{p'} .$$

Not suitable for "stability" analysis.

However: If

$$p(x) \le q(x) \le p(x) + \varepsilon \text{ a.e. in } \Omega. \tag{6}$$

$\rho_q(f) \le 1$, then

$$\rho_p(f) \le \varepsilon |\Omega| + \varepsilon^{-\varepsilon}.$$

Edmunds-Lang-Nekvinda, 2009

This is the substitute of Hölder's inequality for the stability analysis.

$(p_i)_i \subset C(\overline{\Omega})$ p_i increases uniformly to q

$inf_{\Omega} q > n,$

$$(p_i)_i \subset C(\overline{\Omega}) \;\; p_i \text{ increases uniformly to } q$$

$$inf_{\Omega} q > n,$$

Then, there exists a solution to the q-eigenvalue problem,

$$(\lambda_q, u_q)$$

such that

$$u_{p_i} \rightharpoonup u_q \;\text{ in }\; W_0^{1,q}(\Omega)$$

and

$$(p_i)_i \subset C(\overline{\Omega}) \;\; p_i \text{ increases uniformly to } \; q$$

$$inf_{\Omega} q > n,$$

Then, there exists a solution to the q-eigenvalue problem,

$$(\lambda_q, u_q)$$

such that

$$u_{p_i} \rightharpoonup u_q \;\; \text{in} \;\; W_0^{1,q}(\Omega)$$

and

$$u_{p_i} \to u_q \;\; \text{in} \;\; L^q(\Omega)$$

Moreover:

$$(p_i)_i \subset C(\overline{\Omega}) \;\; p_i \text{ increases uniformly to } \; q$$

$$inf_{\Omega} q > n,$$

Then, there exists a solution to the q-eigenvalue problem,

$$(\lambda_q, u_q)$$

such that

$$u_{p_i} \rightharpoonup u_q \;\text{ in }\; W_0^{1,q}(\Omega)$$

and

$$u_{p_i} \to u_q \;\text{ in }\; L^q(\Omega)$$

Moreover:

$$\frac{q_-}{q_+} L \leq \frac{\int_\Omega |\nabla u|^q \, dx}{\int_\Omega |u|^q \, dx} \leq L, \tag{7}$$

and

$$\left(\frac{q_-}{q_+}\right)^2 L \leq \lambda_{q,1} \leq L.$$

$$(p_i)_i \subset C(\overline{\Omega}) \ \ p_i \ \text{ increases uniformly to } \ q$$

$$inf_{\Omega} q > n,$$

Then, there exists a solution to the q-eigenvalue problem,

$$(\lambda_q, u_q)$$

such that

$$u_{p_i} \rightharpoonup u_q \ \text{ in } \ W_0^{1,q}(\Omega)$$

and

$$u_{p_i} \to u_q \ \text{ in } \ L^q(\Omega)$$

Moreover:

$$\frac{q_-}{q_+} L \le \frac{\int_\Omega |\nabla u|^q \, dx}{\int_\Omega |u|^q \, dx} \le L, \tag{7}$$

and

$$\left(\frac{q_-}{q_+}\right)^2 L \le \lambda_{q,1} \le L.$$

Similar result for (p_i) decreasing uniformly to p.

For

$$p, q \geq 2$$

the convergence is actually strong in the corresponding Sobolev Space.

Remarks

The norm on $W_0^{1,p(\cdot)}(\Omega)$,

$$u \longrightarrow |||\nabla u|||_{p(\cdot)} = \|u\|_{1,p(\cdot)}$$

is Fréchet differentiable with derivative given by

$$(\text{grad}\, |||f|||_{1,p(\cdot)})(x) = \frac{p(x)\, |||\nabla f|||_{p(\cdot)}^{-p(x)}\, ||\nabla f(x)||^{p(x)-1}\, \mathrm{sgn}\ \nabla f(x)}{\int\limits_{\Omega} p(x)\, |||\nabla f|||_{p(\cdot)}^{-p(x)-1}\, |\nabla f(x)|^{p(x)}\, dx}.$$

Remarks

The norm on $W_0^{1,p(\cdot)}(\Omega)$,

$$u \longrightarrow |||\nabla u|||_{p(\cdot)} = \|u\|_{1,p(\cdot)}$$

is Fréchet differentiable with derivative given by

$$(\text{grad}\,|||f|||_{1,p(\cdot)})(x) = \frac{p(x)\,|||\nabla f|||_{p(\cdot)}^{-p(x)}\,||\nabla f(x)||^{p(x)-1}\,\mathrm{sgn}\,\nabla f(x)}{\int\limits_{\Omega} p(x)\,|||\nabla f|||_{p(\cdot)}^{-p(x)-1}\,|\nabla f(x)|^{p(x)}\,dx}.$$

Dinca-Mathei (2009)

Remarks

The norm on $W_0^{1,p(\cdot)}(\Omega)$,

$$u \longrightarrow |||\nabla u|||_{p(\cdot)} = \|u\|_{1,p(\cdot)}$$

is Fréchet differentiable with derivative given by

$$(\text{grad}\, |||f|||_{1,p(\cdot)})(x) = \frac{p(x)\, |||\nabla f|||_{p(\cdot)}^{-p(x)}\, ||\nabla f(x)||^{p(x)-1}\, \mathrm{sgn}\ \nabla f(x)}{\int\limits_{\Omega} p(x)\, |||\nabla f|||_{p(\cdot)}^{-p(x)-1}\, |\nabla f(x)|^{p(x)}\, dx}.$$

Dinca-Mathei (2009) Therefore:

$$\int_{\Omega} \text{grad}\left(\|u_0\|_{1,p(\cdot)}\right)(x)\nabla h(x)\, dx = \lambda_p \int_{\Omega} \text{grad}\left(\|u_0\|_{p(\cdot)}\right)(x) h(x)\, dx,$$

with $\lambda_p = \frac{1}{||E||}$.

Remarks

The norm on $W_0^{1,p(\cdot)}(\Omega)$,

$$u \longrightarrow |||\nabla u|||_{p(\cdot)} = \|u\|_{1,p(\cdot)}$$

is Fréchet differentiable with derivative given by

$$(\text{grad}\,|||f|||_{1,p(\cdot)})(x) = \frac{p(x)\,|||\nabla f|||_{p(\cdot)}^{-p(x)}\,||\nabla f(x)||^{p(x)-1}\,\mathrm{sgn}\,\nabla f(x)}{\int\limits_{\Omega} p(x)\,|||\nabla f|||_{p(\cdot)}^{-p(x)-1}\,|\nabla f(x)|^{p(x)}\,dx}.$$

Dinca-Mathei (2009) Therefore:

$$\int_{\Omega} \text{grad}\left(\|u_0\|_{1,p(\cdot)}\right)(x)\nabla h(x)\,dx = \lambda_p \int_{\Omega} \text{grad}\left(\|u_0\|_{p(\cdot)}\right)(x) h(x)\,dx,$$

with $\lambda_p = \frac{1}{\|E\|}$.
is the Euler-Lagrange equation corresponding to the maximal functions of the Sobolev embedding!!!!

Remarks

The norm on $W_0^{1,p(\cdot)}(\Omega)$,

$$u \longrightarrow |||\nabla u|||_{p(\cdot)} = \|u\|_{1,p(\cdot)}$$

is Fréchet differentiable with derivative given by

$$(\text{grad}\,|||f|||_{1,p(\cdot)})(x) = \frac{p(x)\,|||\nabla f|||_{p(\cdot)}^{-p(x)}\,||\nabla f(x)||^{p(x)-1}\,\mathrm{sgn}\,\nabla f(x)}{\int\limits_{\Omega} p(x)\,|||\nabla f|||_{p(\cdot)}^{-p(x)-1}\,|\nabla f(x)|^{p(x)}\,dx}.$$

Dinca-Mathei (2009) Therefore:

$$\int_{\Omega} \text{grad}\left(\|u_0\|_{1,p(\cdot)}\right)(x)\nabla h(x)\,dx = \lambda_p \int_{\Omega} \text{grad}\left(\|u_0\|_{p(\cdot)}\right)(x)h(x)\,dx,$$

with $\lambda_p = \frac{1}{||E||}$.
is the Euler-Lagrange equation corresponding to the maximal functions of the Sobolev embedding!!!! Bennewitz (2007), Lindqvist-Francina (2013)

Open problems

Open problems

Strong convergence in general?

Open problems

Strong convergence in general?
Uniqueness of positive eigenfunction?

Open problems

Strong convergence in general?
Uniqueness of positive eigenfunction? Implied by the uniform convexity of

$$W_0^{1,p}(\Omega)$$

with respect to the norm

$$u \to \||\nabla u|\|_p.$$

Open problems

Strong convergence in general?

Uniqueness of positive eigenfunction? Implied by the uniform convexity of

$$W_0^{1,p}(\Omega)$$

with respect to the norm

$$u \to |||\nabla u|||_p.$$

Other boundary conditions? (e.g. Neumann, Robin??)

MUITO OBRIGADO!

MUITO OBRIGADO!

THANKS!