Approximation Problems in the Framework of New Nonstandard Banach Function Spaces

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AMS-EMS-SPM Meeting in Porto, Portugal
July 10-13, 2015

Outline of the talk

- Two-weighted Bernstein-Zygmund-Nikol'skii type inequalities in classical Lebesgue spaces, applications
- Weighted inequalities for trigonometric polynomials in Iwaniec-Sbordone and grand variable exponent Lebesgue spaces, applications
- Weighted inequalities in variable exponent Lebesgue spaces and approximation problems
- The conjugate functions in $L^{p(\cdot)}$ spaces when $p_- = \inf p(x) = 1$. Invariant classes

Two-weighted estimates in Lebesgue Spaces

- For the further use, we need to make the following definitions:

$$\mathbb{T} = [-\pi, \pi], \quad L^p(\mathbb{T}, w) := \{f : \|fw\|_p < +\infty\}, \quad \|f\|_{p,w} := \|fw\|_p,$$

$$\mathbb{W}_w^{p,r} = \{f : \|f\|_{p,w} + \|f^{(r)}\|_{p,w} < \infty\}, \quad r > 0.$$

For $f \in L^p(\mathbb{T}, w)$ the structural characteristic is defined via the Steklov means:

$$\Omega(f, \delta)_{p,w} = \sup_{0<h\le\delta} \left\| \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt - f(x) \right\|_{p,w}$$

- We need also the notion of best approximations by trigonometric polynomials $$E_n(f)_{p,w} = \inf_{T_n} \|f - T_n\|_{p,w},$$ where T_n are trigonometric polynomials of degree $\le n$.

- Pair of weights $(v, w) \in \mathbb{A}_{p,q}$, $1 < p \le q < \infty$ if

$$\sup_I \left(\frac{1}{|I|} \int_I v^q(x)dx \right)^{1/q} \left(\frac{1}{|I|} \int_I w^{-p'}(x)dx \right)^{1/p'} < +\infty.$$

We set $\mathbb{A}_p := \mathbb{A}_{p,p}$.

Theorem 1.1. Let $1 < p \leq q < \infty$, $(v, w) \in \mathbb{A}_{p,q}$. Then for arbitrary trigonometric polynomial T_n we have

$$\| T_n' \|_{q,v} \leq c n^{1+1/p-1/q} \| T_n \|_{p,w}$$

with a constant independent of n and T_n.

Theorem 1.2. Let $1 < p \leq q < \infty$, $(v, w) \in \mathbb{A}_{p,q}$ and, let $\gamma = \min(2, q)$. Let

$$\sum_{j=1}^{\infty} \nu^{\gamma(1+1/p-1/q)-1} E_\nu(f)_{p,w} < \infty,$$

then $f \in \mathbb{W}_w^{q,1}$ and the following inequality holds

$$\Omega\left(f', \frac{1}{n}\right)_{q,v} \leq c\Bigg(\frac{1}{n^2}\left(\sum_{\nu=1}^{n} \nu^{\gamma(3+1/p-1/q)-1} E_{\nu-1}^{\gamma}(f)_{p,w}\right)^{1/\gamma} + c\left(\sum_{\nu=n+1}^{\infty} \nu^{\gamma(1+1/p-1/q)-1} E_\nu(f)_{p,w}\right)^{1/\gamma}\Bigg).$$

- The last inequality gives more precise estimate than the one due to M. Timan in the case $v \equiv w \equiv 1$ (see e. g. R. A. Devore, G. G. Lorentz, Constructive Approximation, Springer 1993, p. 210)

Corollary. Let

$$E_\nu(f)_{p,w} = O\left(\frac{1}{\nu^{3+1/p-1/q}}\right)$$

then $f \in \mathbb{W}_w^{q,1}$ and

$$\Omega(f', \delta)_{q,w} = O\left(\delta^2 \left(\ln \frac{1}{\delta}\right)^{\frac{1}{\gamma}}\right).$$

For a Borel set e we set $ve = \int\limits_e v(x)dx$. Let $I(x,r) = (x-r, x+r) \cap \mathbb{T}$, $x \in \mathbb{T}$.

Theorem 1.3. Let $1 < p < q < \infty$, $0 < \alpha < 1$. Let

$$\sup_{I(x,r)} (vI(x,r))^{1/q} \left(\int\limits_{\mathbb{T}\setminus I(x,r)} \frac{w^{1-p'}(y)dx}{|x-y|^{(1-\alpha)p'}} \right)^{1/p'} < +\infty$$

and $w \in A_p$. Then for arbitrary trigonometric polynomial of degree n and $\lambda > 0$ the following inequality holds

$$\lambda^q v\{x : \ |T_n(x)| > \lambda\} \leq cn^{\alpha} \|T_n\|_{L^p(\mathbb{T},w)}$$

Theorem 1.4. Let $1 < p < q < \infty$, $0 < \alpha < 1$. Let together with the conditions of Theorem 1.3 the condition

$$\sup_{I(x,r)} (w^{1-p'} I(x,r))^{1/p'} \left(\int\limits_{\mathbb{T}\setminus I(x,r)} \frac{v^q(y)dx}{|x-y|^{(1-\alpha)q}} \right)^{1/q} < \infty$$

is fulfilled. Then

$$\|T_n\|_{L^q(\mathbb{T},v)} \leq cn^{\alpha} \|T_n\|_{L^p(\mathbb{T},w)}$$

Weighted grand Lebesgue spaces and variable grand Lebesgue spaces

Weighted Iwaniec-Sbordone spaces. Let $1 < p < \infty$, $\theta > 0$ and let w be a weight function.

$$L_w^{p),\theta}(\mathbb{T}) = \{f : \|f\|_{L_w^{p),\theta}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-1}} \left(\int_{\mathbb{T}} |f(x)|^{p-\varepsilon} w(x) dx \right)^{\frac{1}{p-\varepsilon}} < \infty$$

$$L^{p),\theta}(\mathbb{T}, w) = \{f : \|f\|_{p),\theta,w} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-1}} \left(\int_{\mathbb{T}} |f(x)w(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty$$

- It is known (see A. Fiorenza, B. Gupta and P. Jain. Studia Math. 188(2008), No 2, 123-133.), that the equivalence $f \in L_w^p \Leftrightarrow fw^{\frac{1}{p}} \in L^p$ fails in grand Lebesgue spaces.
- We get the difference spaces when taking a weight as measure or as a factor.

In the sequel A_p denotes the class of Muckenhoupt weights defined as

$$\sup_I \left(\frac{1}{|I|} \int_I w(x)dx \right) \left(\frac{1}{|I|} \int_I w^{1-p'}(x)dx \right)^{p-1} < +\infty.$$

- For the space $L_w^{p),\theta}(\mathbb{T})$ an one-weighted version of Bernstein-Zygmund-Nikol'skii inequality holds

Theorem 2.1 Let $1 < p \leq q < \infty$, $\theta > 0$, $\theta_1 = \theta q/p$. Let $w \in A_{1+q/p'}$. Then for arbitrary trigonometric polynomial we have

$$\left\| T_n^{(r)} w^{\frac{1}{p}-\frac{1}{q}} \right\|_{L_w^{q),\theta_1}} \leq c n^{r+1/p-1/q} \| T_n \|_{L_w^{p),\theta}} \quad r = 0, 1, 2, \ldots$$

with a positive constant c independent of n and T_n.

Theorem 2.2. Let $1 < p < q < \infty$, $\theta > 0$ and $0 < \alpha < 1$. Let for the Borel measure μ of $I(x, r)$ the condition

$$\mu I(x, r) \leq c r^{q\left(\frac{1}{p} - \alpha\right)}$$

is fulfilled with a constant c independent of $I(x, r)$. Then the inequality

$$\| T_n \|_{L^{q),\theta \frac{q}{p}}(\mathbb{T}, d\mu)} \leq c_1 n^{\alpha} \| T_n \|_{L^{p),\theta}(\mathbb{T}, dx)}$$

holds where a constant c_1 does not depend on T_n and n.

- In approximation problems more convenient is the case when w in the norm participates as a factor. For this case the following statement is true:

Theorem 2.3 Let $1 < p \leq q < \infty$, $\theta > 0$, $\theta_1 = \theta q/p$. Let $w \in \mathbb{A}_{p,q}$. Then for α-order fractional derivative of trigonometric polynomials the following estimate holds

$$\| T_n^{(\alpha)} w \|_{q),\theta_1} \leq cn^{\alpha+1/p-1/q} \| T_n w \|_{p),\theta}, \quad \alpha \geq 0$$

with a constant $c > 0$ independent of n and T_n.

- The following assertion is a refined form of Bernstein type inequality.

Theorem 2.4 Let $1 < p < \infty$, $\theta > 0$ and $w \in \mathbb{A}_p$. Then

$$\| T_n^{(\alpha)} w \|_{p),\theta} \leq cn^{\alpha} \Omega \left(T_n, \frac{1}{n} \right)_{p),\theta,w} \quad \alpha > 0$$

with a constant $c > 0$ independent of n and T_n.

- The crusial role in the proof of last two theorems plays the following boundedness theorem for Riesz potentials.

Theorem 2.4 Let $1 < p < \infty$, $\theta > 0$, $\theta_1 = \theta q/p$ and, let $q = \frac{p}{1-p\alpha} > 0$. If $w \in \mathbb{A}_{p,q}$, then

$$\|I_\alpha fw\|_{q),\theta_1} \leq c\|fw\|_{p),\theta}$$

with a constant independent of f.

The weighted grand Lebesgue spaces are non-reflexive, non-separable and non-rearrangement spaces. The approximable by trigonometric polynomials subspace of $L^p(\mathbb{T}, w)$ is the set of functions defined by the condition

$$\lim_{\varepsilon\to 0} \varepsilon^{\theta}\|fw\|_{p-\varepsilon} = 0.$$

This is the closure of $L^p(\mathbb{T}, w)$ by the norm of $L^{p),\theta}(\mathbb{T}, w)$. This subspace is denoted by $\widetilde{L}^{p),\theta}(\mathbb{T}, w)$.
For the functions $f \in \widetilde{L}^{p),\theta}(\mathbb{T}, w)$ we set

$$\Omega(f, \delta)_{p),\theta,w} = \sup_{0<h<\delta} \left\| \frac{1}{h} \int_{x-h}^{x+h} f(t)dt - f(x) \right\|_{p),\theta,w}.$$

It is easy to see that

$$\lim_{\delta\to 0} \Omega(f, \delta)_{p),\theta,w} = 0$$

for every $f \in \widetilde{L}_w^{p),\theta}$.

- The following two-sided estimates of moduli of smoothness is essential in approximation theory.

Theorem 2.5 Let $1 < p < \infty$, $\theta > 0$ and let $w \in \mathbb{A}_p$. Then

$$\Omega\left(f, \frac{1}{n}\right)_{p),\theta,w} \approx \|f - S_n\|_{p),\theta,w} + n^{-2}\|S_n''\|_{p),\theta,w}.$$

$a \approx b$ means that there exist c_1 and c_2 such that $c_1 b \leq a \leq c_2 b$.

- On the base of this two-sided estimates we prove the direct and inverse theorem (in Bernstein's terminology) of constructive theory of functions.

Theorem 2.6 Let $1 < p < \infty$, $w \in \mathbb{A}_p$. Them for $f \in \widetilde{\mathbb{W}}_w^{p),\theta,r}$, $r \geq 0$ we have the Jackson type inequality

$$E_n(f)_{p),w,\theta} \leq \frac{c}{(n+1)^r}\Omega\left(f^{(r)}, \frac{1}{n}\right)_{p),\theta,w}$$

with a constant independent of n and f.

- The inverse statement sounds as

Theorem 2.7 Let $1 < p \leq q < \infty$ and $w \in \mathbb{A}_{p,q}$. Let $\theta > 0$ and $\theta_1 \geq \theta \cdot \frac{q}{p}$, and let for $f \in \widetilde{L}^{p),\theta}(\mathbb{T}, w)$ the series

$$\sum_{\nu=1}^{\infty} \nu^{\alpha+1/p-1/q-1} E_\nu(f)_{p),\theta,w} < \infty, \quad \alpha > 0.$$

Then $f \in \widetilde{\mathbb{W}}_w^{q),\theta_1,\alpha}$ and

$$\Omega\left(f^{(\alpha)}, \frac{1}{n}\right)_{q),\theta_1,w} \leq c\bigg(\frac{1}{n^2}\sum_{\nu=1}^{n} \nu^{\frac{1}{p}-\frac{1}{q}+1+\alpha} E_{\nu-1}(f)_{p),\theta,w}$$

$$+ \sum_{\nu=n+1}^{\infty} \nu^{\alpha+1/p-1/q-1} E_\nu(f)_{p),\theta,w}\bigg).$$

with a constant c is independent of f and n.

- This estimate on a certain subclass of functions is unimprovable.

Variable exponent Lebesgue spaces

Let p be a measurable function 2π-periodic and continuous on the real line with local log-continuity condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| < \frac{1}{2}$$

for which $p_- := \min_{\mathbb{T}} p(x) > 1$.

The class of such exponents is denoted by $\mathcal{P}^{\log}$.

$$L^{p(\cdot)}(\mathbb{T}, w) = \inf\{\lambda > 0 : \int_{\mathbb{T}} \left|\frac{f(x)w(x)}{\lambda}\right|^{p(x)} dx \leq 1\}.$$

The space $L^{p(\cdot)}(\mathbb{T}, w)$ is a Banach function space.

- **Class of weights.** For given p and q from $\mathcal{P}^{\log}$ the class of weights $\mathbb{A}_{p(\cdot),q(\cdot)}$ is defined by the condition

$$\sup_I \|\chi_I w\|_{q(\cdot)} \cdot \|\chi_I w^{-1}\|_{p'(\cdot)} < c|I|^{1-s}.$$

Here $p'(x) = \frac{p(x)}{p(x)-1}$, $\frac{1}{p(x)} - \frac{1}{q(x)} \equiv s \geq 0$, $x \in \mathbb{T}$.

Theorem 3.1. Let $p \in \mathcal{P}^{\log}$, $\frac{1}{p(x)} - \frac{1}{q(x)} = s \geq 0$ for $x \in \mathbb{T}$. Let $p_+ := \max p(x) < \frac{1}{s}$ and $w \in \mathbb{A}_{p(\cdot),q(\cdot)}$. Then the following Bernstein-Zygmund-Nikol'skii type inequality

$$\|T_n^{(\alpha)}\|_{q(\cdot),w} \leq cn^{\alpha+1/p-1/q}\|T_n\|_{p(\cdot),w} \quad \alpha \geq 0$$

holds with a constant c independent of T_n and n.

- To explore the approximation problems in variable exponent weighted Lebesgue spaces we introduce the appropriate K-functional. Let
  $\mathbb{W}_w^{p(\cdot),2} = \{g : \ \|g\|_{p(\cdot),w} + \|g''\|_{p(\cdot),w} < +\infty\}$
  $$K_2(f,t,L^{p(\cdot)}(\mathbb{T},w),\mathbb{W}_w^{p(\cdot),2}) = \inf_{g \in \mathbb{W}_w^{p(\cdot),2}} \{\|f-g\|_{p(\cdot),w} + t^2\|g''\|_{p(\cdot),w}\}.$$

Theorem 3.2. Let $p \in \mathcal{P}^{\log}$, $w \in \mathbb{A}_{p(\cdot)}$. Then

$$K_2(f,t,L^{p(\cdot)}(\mathbb{T},w),\mathbb{W}_w^{p(\cdot),2}) \approx \Omega\left(f,\frac{1}{n}\right)_{p(\cdot),w}.$$

- On the base of this theorem we establish the direct and inverse estimates in variable exponent weighted Lebesgue spaces.

Theorem 3.3 (Refined Jackson's type ineqaulity) Let $p \in \mathcal{P}^{\log}$, $w \in A_{p(\cdot)}$, $\beta = \max(2, p^+)$. Then for $f \in L^{p(\cdot)}(\mathbb{T}, w)$ the estimate holds

$$\frac{1}{n^2}\left\{\sum_{\nu=1}^{n} \nu^{2\beta-1} E_{\nu-1}^{\beta}(f)_{p(\cdot),w}\right\}^{\frac{1}{\beta}} \leq c\Omega\left(f, \frac{1}{n}\right)_{p(\cdot),w}$$

with a constant independent of n and f.

- This estimate improves the estimate of I. Sharapudinov (the case $w \equiv 1$) $\quad E_n(f)_{p(\cdot)} \leq c\Omega\left(f, \frac{1}{n}\right)_{p(\cdot)}$

The inverse result is contained in the following

Theorem 3.4 Let $p \in \mathcal{P}^{\log}$, $w \in A_{p(\cdot)}$, $\gamma = \min(2, p_-)$. Then for $f \in L^{p(\cdot)}(\mathbb{T}, w)$ the following estimate holds

$$\Omega\left(f, \frac{1}{n}\right)_{p(\cdot),w} \leq \frac{c}{n^2}\left\{\sum_{\nu=1}^{n} \nu^{2\gamma-1} E_{\nu-1}^{\gamma}(f)_{p(\cdot),w}\right\}^{1/\gamma}$$

- This estimate is essentially better than the estimate (even for $w \equiv 1$) (I. Sharapudinov, Chaichenko etc):

$$\Omega\left(f, \frac{1}{n}\right)_{p(\cdot)} \leq \frac{c}{n^2} \cdot \sum_{\nu=1}^{n} \nu E_{\nu-1}(f)_{p(\cdot)}.$$

Conjugate operators in $L^{p(\cdot)}$ when $p_{-} = 1$

In this section we study the question: under what conditions the conjugate function $\widetilde{f}$ belongs to $L^{p(\cdot)}$ and to find in terms of $\Omega(f,\delta)_{p(\cdot)}$ an invariant class with respect to conjugate operator, to give the applications to the boundary problems for analytic and harmonic functions.

Denote by $\mathcal{P}_0^{\log}$ the class of 2π-periodic exponents with local log-continuity condition and the condition $\min\limits_{\mathbb{T}} p(x) = 1$.

For this case we have the following results.

Theorem 4.1. Let $p \in \mathcal{P}_0^{\log}$ and let for $f \in L^{p(\cdot)}(\mathbb{T})$ the integral

$$\int\limits_0^{\delta_0} \frac{\Omega(f,t)_{p(\cdot)}}{t} < +\infty.$$

Then $\widetilde{f} \in L^{p(\cdot)}$ and the estimate

$$\Omega(\widetilde{f},\delta)_{p(\cdot)} \leq c \left(\int\limits_0^{\delta} \frac{\Omega(f,t)_{p(\cdot)}}{t} + \delta^2 \int\limits_{\delta}^{\delta_0} \frac{\Omega(f,t)_{p(\cdot)}}{t^3} dt \right), \quad 0 < \delta < \delta_0$$

holds with a constant c independent of f and δ.

Theorem 4.2. Let $p \in \mathcal{P}_0^{\log}$ and let for $f \in L^{p(\cdot)}(\mathbb{T})$ and $r \in \mathbb{N}$ the integral $\int\limits_0^{\delta_0} \frac{\Omega(f,t)}{t^{r+1}} dt < +\infty$. Then $(\widetilde{f})^{(r)} \in L^{p(\cdot)}$ and we have

$$\Omega((\widetilde{f})^{(r)}, \delta) \leq c \left(\int\limits_0^{\delta} \frac{\Omega(f,t)_{p(\cdot)}}{t^{r+1}} dt + \delta^{2r} \int\limits_0^{\delta_0} \frac{\Omega(f,t)_{p(\cdot)}}{t^{2r+1}} dt \right).$$

- Let for $k = 0, 1, \ldots$ define the subclass of $L^{p(\cdot)}$

$$V_k = \left\{ f \in L^{p(\cdot)} : \int\limits_0^{\delta_0} \frac{\Omega(f,t)}{t} \left(\ln \frac{\pi}{t} \right)^k dt < +\infty \right\}.$$

From Theorem 4.1 it is clear that

$$f \in V_k \Longrightarrow \widetilde{f} \in V_{k-1}.$$

Thus we have that the class $V = \bigcap\limits_{k=0}^{\infty} V_k$ is invariant with respect to the conjugate operator.

$$f \in V \Longrightarrow \widetilde{f} \in V$$

Theorem 4.3. The functions from the Dini class i. e.

$$D := \{g \in C : \int_0^{\delta_0} \frac{\omega(g,\delta)}{\delta} d\delta < +\infty\}$$

are pointwise multipliers for V_0 i. e.

$$f \in V_0,\ g \in D \Longrightarrow fg \in V_0.$$

- One of the tool of proofs is an extension of Bernstein-Nikol'skii-Stechkin result. We prove the following

Theorem 4.4 Let $p \in \mathcal{P}_0^{\log}$. Then for arbitrary trigonometric polynomial T_n the following inequality

$$\|T_n'\|_{p(\cdot)} \leq cn\Omega\left(T_n, \frac{1}{n}\right)_{p(\cdot)}$$

holds with a constant c independent of T_n and n.

On weighted Bernstein type inequality in grand variable exponent Lebesgue spaces

- These spaces unify two nonstandard Banach function spaces: variable exponent Lebesgue and grand Lebesgue spaces.

Let 2π-periodic continuous on the real line function $p \in \mathcal{P}^{\log}$, $\theta > 0$. Then by $L^{p(\cdot),\theta}(\mathbb{T})$ is denoted the set of 2π-periodic functions for which

$$\|f\|_{L^{p(\cdot),\theta}} = \sup_{0<\varepsilon<p_- -1} \varepsilon^{\frac{\theta}{p_- -1}} \|f\|_{L^{p(\cdot)-\varepsilon}} < +\infty.$$

These spaces are non-reflexive, non-separable and non-rearrangement invariant spaces when p is a constant.

- In these spaces the following extension of Bernstein-Ditzian-Totik inequality is true.

Theorem 5.1 For any $r \in \mathbb{N}$ and trigonometric polynomial T_n of degree less than or equal to n, the inequality

$$\|\sin^r t T_n'(t)\|_{L^{p(\cdot),\theta}} \leq cn \|\sin^r t T_n(t)\|_{L^{p(\cdot),\theta}}$$

holds with a constant c independent of T_n.

Theorem 5.1 For any $r \in \mathbb{N}$ and trigonometric polynomial T_n of degree less than or equal to n, the inequality

$$\|\sin^r tT_n'(t)\|_{L^{p(\cdot),\theta}} \leq cn\|\sin^r tT_n(t)\|_{L^{p(\cdot),\theta}} \quad (1)$$

holds with a constant c independent of T_n.

- It should be emphasized that even for constant p, the special weights inside of norms in (1) is more general than the Mackenhoupt weights of the same type.

References

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