Maximally modulated Hilbert transform and its applications to pseudodifferential operators on variable Lebesgue spaces

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Variable Lebesgue spaces

Let $p\colon \mathbb{R} \to [1, \infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R})$ we denote the set of all complex-valued functions f on $\mathbb{R}$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$.
This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \le 1 \right\}.$$

It is easy to see that if p is constant, then $L^{p(\cdot)}(\mathbb{R})$ is nothing but the standard Lebesgue space $L^p(\mathbb{R})$. The space $L^{p(\cdot)}(\mathbb{R})$ is referred to as a *variable Lebesgue space*.
Put

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}} p(x).$$

We will assume that

$$1 < p_-, \quad p_+ < \infty.$$

Hilbert transform (HT) and maximally modulated HT

For a bounded compactly supported function $f : \mathbb{R} \to \mathbb{C}$, consider its *Hilbert transform* (HT) defined by

$$(Hf)(x) := \frac{1}{\pi}\text{v.p.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}\setminus I(x,\varepsilon)} \frac{f(y)}{x-y}\, dy, \quad x \in \mathbb{R},$$

where $I(x,\varepsilon) := (x-\varepsilon, x+\varepsilon)$.

For $\alpha \in \mathbb{R}$, consider the following multiplication operator

$$(\mathcal{M}_\alpha f)(x) := e^{-\alpha i x} f(x), \quad x \in \mathbb{R}.$$

The maximally modulated Hilbert transform (MMHT) is defined by

$$(\mathcal{C} f)(x) := \sup_{\alpha \in \mathbb{R}} \left|(H(\mathcal{M}_\alpha f))(x)\right|, \quad x \in \mathbb{R}.$$

The Hardy-Littlewood maximal operator

Let $1 \leq q < \infty$. Given $f \in L^q_{\mathrm{loc}}(\mathbb{R})$, the *q-th maximal operator* is defined by

$$(M_q f)(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y)|^q dy \right)^{1/q}, \quad x \in \mathbb{R},$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ containing x.

For $q = 1$ this is the usual Hardy-Littlewood maximal operator. We will use the standard notation

$$M = M_1$$

Globally log-Hölder continuous variable exponents

Let $LH(\mathbb{R})$ denote the class of variable exponents $p : \mathbb{R} \to [1, \infty]$ satisfying

1. $$1 < p_-, \quad p_+ < \infty;$$

2. $$|p(x) - p(y)| \le \frac{c}{\log(1/|x-y|)} \quad (x, y \in \mathbb{R}, \quad |x-y| < 1/2);$$

3. $$|p(x) - p(y)| \le \frac{c}{\log(e+|x|)} \quad (x, y \in \mathbb{R}, \quad |y| \ge |x|).$$

The Hardy-Littlewood maximal function on variable Lebesgue spaces (very incomplete history)

Let $\mathcal{B}_M(\mathbb{R})$ be the class of variable exponents satisfying

$$1 < p_-, \quad p_+ < \infty$$

and such that M is bounded on $L^{p(\cdot)}(\mathbb{R})$.

Theorem (Diening, 02; Cruz-Uribe, Fiorenza, Neugebauer, 03)

$LH(\mathbb{R}) \subset \mathcal{B}_M(\mathbb{R})$.

Theorem (Nekvinda, 2004; Lerner, 2005)

$LH(\mathbb{R})$ is a proper subset of $\mathcal{B}_M(\mathbb{R})$.

Theorem (Deining, 2005)

$p \in \mathcal{B}_M(\mathbb{R}) \Longleftrightarrow p' \in \mathcal{B}_M(\mathbb{R})$.

Boundedness of the MMHT on standard Lebesgue spaces

Let $S_n f$ denote the n-th partial sum of th Fourier series of $f \in L^1(\mathbb{T})$ and

$$(\mathcal{M}f)(t) := \sup_{n \geq 0} |(S_n f)(t)|, \quad t \in \mathbb{T}.$$

- Boundedness of $\mathcal{M}$ on $L^2(\mathbb{T})$ (Carleson, 1966)
- Boundedness of $\mathcal{M}$ on $L^p(\mathbb{T})$, $1 < p < \infty$ (Hunt, 1968)
- Boundedness of the MMHT on $L^p(\mathbb{R})$, $1 < p < \infty$ (Kenig, Tomas, 1980)
- Independent direct proof of the boundedness of the MMHT on $L^p(\mathbb{R})$, $1 < p < \infty$ (Lacey, Thiele, 2000)

Boundedness of the MMHT on variable Lebesgue spaces

Theorem (A.K., 2014)

If $p \in \mathcal{B}_M(\mathbb{R})$, then the maximally modulated Hilbert transform $\mathcal{C}$ extends to a bounded operator on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$.

The below proof also works for general Banach function spaces $X(\mathbb{R})$ in the sense of W. Luxemburg under the assumption that

- $X(\mathbb{R})$ is separable;
- M is bounded on $X(\mathbb{R})$;
- M is bounded the associate space $X'(\mathbb{R})$.

Self-improving property

If $1 < q < \infty$, then from the Hölder inequality one can immediately get that

$$(Mf)(x) \le (M_q f)(x) \quad (x \in \mathbb{R}).$$

Thus, for every $1 < q < \infty$,

$$\|Mf\|_{X(\mathbb{R})} \le \|M_q f\|_{X(\mathbb{R})}.$$

Theorem (Lerner-Pérez, 2007)

Let $X(\mathbb{R})$ be a Banach function space. Then M is bounded on $X(\mathbb{R})$ if and only if M_q is bounded on $X(\mathbb{R})$ for some $q \in (1, \infty)$.

One class of functions

Let $S_0(\mathbb{R})$ be the space of all measurable functions f on $\mathbb{R}$ such that

$$|\{x \in \mathbb{R} : |f(x)| > \lambda\}| < \infty$$

for any $\lambda > 0$.

Chebyshev's inequality

$$|\{x \in \mathbb{R} : |f(x)| > \lambda\}| \leq \frac{1}{\lambda^q} \int_{\mathbb{R}} |f(x)|^q \, dx$$

holds for every $q \in (0, \infty)$ and $\lambda > 0$. In particular, it implies that

$$\bigcup_{q \in (0,\infty)} L^q(\mathbb{R}) \subset S_0(\mathbb{R}).$$

The Fefferman-Stein inequality for Banach function spaces

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. For an interval $Q \subset \mathbb{R}$, put

$$f_Q := \frac{1}{|Q|} \int_Q f(x)dx.$$

The Fefferman-Stein sharp maximal operator $f \mapsto f^{\#}$ is defined by

$$f^{\#}(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \quad (x \in \mathbb{R}),$$

where the supremum is taken over all intervals Q containing x.

Theorem (Lerner, 2010)

Let M be bounded on a Banach function space $X(\mathbb{R})$. Then M is bounded on its associate space $X'(\mathbb{R})$ if and only if there exists a constant $C_{\#} > 0$ such that, for all $f \in S_0(\mathbb{R})$,

$$\|f\|_{X(\mathbb{R})} \le C_{\#} \|f^{\#}\|_{X(\mathbb{R})}.$$

The crucial pointwise estimate

Theorem (Grafakos, Martell, Soria 2005)

If $q \in (1, \infty)$*, then there exists a constant* $C_q > 0$ *such that for every* $f \in C_0^\infty(\mathbb{R})$*,*

$$(\mathcal{C}f)^\#(x) \le C_q (M_q f)(x) \quad \textit{for a.e.} \quad x \in \mathbb{R}.$$

Proof of the boundedness result

Let $f \in C_0^\infty(\mathbb{R})$. Then $\mathcal{C}f \in S_0(\mathbb{R})$. By Lerner's theorem,

$$\|\mathcal{C}f\|_{X(\mathbb{R})} \le C_\# \|(\mathcal{C}f)^\#\|_{X(\mathbb{R})}.$$

By the pointwise estimate, for every $q \in (1, \infty)$,

$$(\mathcal{C}f)^\#(x) \le C_q (M_q f)(x) \quad (x \in \mathbb{R}).$$

Hence

$$\|(\mathcal{C}f)^\#\|_{X(\mathbb{R})} \le C_q \|M_q f\|_{X(\mathbb{R})}.$$

On the other hand, since M is bounded on $X(\mathbb{R})$, by the Lerner-Pérez theorem, there is a $q_0 \in (1, \infty)$ and $C'_{q_0} > 0$ such that

$$\|M_{q_0} f\|_{X(\mathbb{R})} \le C'_{q_0} \|f\|_{X(\mathbb{R})}.$$

Thus, for all $f \in C_0^\infty(\mathbb{R})$,

$$\|\mathcal{C}f\|_{X(\mathbb{R})} \le C_\# C_{q_0} C'_{q_0} \|f\|_{X(\mathbb{R})}.$$

It remains to recall that $C_0^\infty(\mathbb{R}^n)$ is dense in $X(\mathbb{R})$ whenever $X(\mathbb{R})$ is separable.

Functions of bounded total variation I

Let a be a complex-valued function of bounded total variation $V(a)$ on $\mathbb{R}$ where

$$V(a) := \sup \sum_{k=1}^{n} |a(x_k) - a(x_{k-1})|$$

and the supremum is taken over all possible partitions

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty, \quad n \in \mathbb{N}.$$

The set $V(\mathbb{R})$ of all continuous from the left functions of bounded total variation on $\mathbb{R}$ is a unital non-separable Banach algebra with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a).$$

Functions of bounded total variation II

By analogy with

$$V(a) = V_{-\infty}^{+\infty}(a),$$

one can define the following total variations of a function $a : \mathbb{R} \to \mathbb{C}$

(a) $V_c^d(a)$ on $[c, d]$,

(b) $V_{-\infty}^c(a)$ on $(-\infty, c]$,

(c) $V_d^{+\infty}(a)$ on $[d, +\infty)$,

taking, respectively, the partitions

(a) $c = x_0 < x_1 < \cdots < x_n = d$,

(b) $-\infty < x_0 < x_1 < \cdots < x_n = c$,

(c) $d = x_0 < x_1 < \cdots < x_n < +\infty$.

Non-regular symbols of pseudodifferential operators

Let $L^\infty(\mathbb{R}, V(\mathbb{R}))$ be the set of functions

$$a : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$$

such that

$$\widehat{a} : x \mapsto a(x, \cdot)$$

is a bounded measurable $V(\mathbb{R})$-valued function on $\mathbb{R}$.

The measurability of $\widehat{a}$ means that the map $\widehat{a} : \mathbb{R} \to V(\mathbb{R})$ possesses the Luzin property: for any compact set $K \subset \mathbb{R}$ and any δ there is a compact set $K_\delta \subset K$ such that $|K \setminus K_\delta| < \delta$ and $\widehat{a}$ is continuous on K_δ.

$L^\infty(\mathbb{R}, V(\mathbb{R}))$ is a unital Banach algebra with the norm

$$\|a\|_{L^\infty(\mathbb{R},V(\mathbb{R}))} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} \|a(x, \cdot)\|_V.$$

Boundedness of PDO on variable Lebesgue spaces

Theorem (A.K., 2014)

If $p \in \mathcal{B}_M(\mathbb{R})$ *and* $a \in L^\infty(\mathbb{R}, V(\mathbb{R}))$*, then the pseudodifferential operator* $a(x, D)$*, defined for the functions* $f \in C_0^\infty(\mathbb{R})$ *by the iterated integral*

$$(a(x,D)f)(x) := \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}} a(x,\lambda)e^{i(x-y)\lambda}f(y)\,dy, \quad x \in \mathbb{R},$$

extends to a bounded linear operator on the space $L^{p(\cdot)}(\mathbb{R})$ *and*

$$\|a(x,D)\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \le 2\|\mathcal{C}\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))}\|a\|_{L^\infty(\mathbb{R},V(\mathbb{R}))}.$$

This result follows from the boundedness of $\mathcal{C}$ on $L^{p(\cdot)}(\mathbb{R})$ and the following pointwise estimate obtained by Yuri Karlovich in 2007:

$$|(a(x,D)f)(x)| \le 2(\mathcal{C}f)(x)\|a(x,\cdot)\|_V \quad \text{for a.e.} \quad x \in \mathbb{R}$$

and every $f \in C_0^\infty(\mathbb{R})$.

Compactness of PDO on variable Lebesgue spaces

Theorem (A.K., 2014)

Suppose $p \in \mathcal{B}_M(\mathbb{R})$. If $a \in L^\infty(\mathbb{R}, V(\mathbb{R}))$ and

(a) $a(x, \pm\infty) = 0$ *for almost all* $x \in \mathbb{R}$;

(b) $\lim\limits_{|x|\to\infty} V(a(x,\cdot)) = 0$;

(c) *for every* $N > 0$,

$$\lim_{L\to+\infty} \operatorname*{ess\,sup}_{|x|\le N} \Big(V_{-\infty}^{-L}(a(x,\cdot)) + V_L^{+\infty}(a(x,\cdot)) \Big) = 0;$$

then the pseudodifferential operator $a(x, D)$ is compact on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$.

This result follows from the corresponding compactness result for standard Lebesgue spaces $L^r(\mathbb{R})$, $1 < r < \infty$, obtained by Yuri Karlovich in 2007 and the following lemma.

Transferring the compactness from standard to variable Lebesgue spaces

Lemma (in this form - A.K., 2014)

Let A be a linear operator bounded on $L^{p(\cdot)}(\mathbb{R})$ whenever $p \in \mathcal{B}_M(\mathbb{R})$. If A is compact on some (equivalently, all) $L^q(\mathbb{R})$, $1 < q < \infty$, then A is compact on $L^{p(\cdot)}(\mathbb{R})$.

Interpolation of boundedness and compactness

Theorem

Let $p_j : \mathbb{R} \to [1, \infty]$, $j = 0, 1$, be a.e. finite measurable functions, and let $p_\theta : \mathbb{R} \to [1, \infty]$ be defined for $\theta \in [0, 1]$ by

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_0(x)} + \frac{1-\theta}{p_1(x)}, \quad x \in \mathbb{R}.$$

Suppose A is a linear operator defined on $L^{p_0(\cdot)}(\mathbb{R}) + L^{p_1(\cdot)}(\mathbb{R})$.

(a) *(follows from Hudzik, Musielak, Urbański, 1980)*
If $A \in \mathcal{B}(L^{p_j(\cdot)}(\mathbb{R}))$ for $j = 0, 1$, then $A \in \mathcal{B}(L^{p_\theta(\cdot)}(\mathbb{R}))$ for all $\theta \in [0, 1]$ and

$$\|A\|_{\mathcal{B}(L^{p_\theta(\cdot)}(\mathbb{R}))} \le 4\|A\|^{\theta}_{\mathcal{B}(L^{p_0(\cdot)}(\mathbb{R}))}\|A\|^{1-\theta}_{\mathcal{B}(L^{p_1(\cdot)}(\mathbb{R}))}.$$

(b) *(follows from Cobos, Kühn, Schonbeck, 1992)*
If $A \in \mathcal{K}(L^{p_0(\cdot)}(\mathbb{R}))$ and $A \in \mathcal{B}(L^{p_1(\cdot)}(\mathbb{R}))$, then $A \in \mathcal{K}(L^{p_\theta(\cdot)}(\mathbb{R}))$ for all $\theta \in (0, 1)$.

Important characterization of the class $\mathcal{B}_M(\mathbb{R})$

We denote by $\mathcal{B}_M^*(\mathbb{R})$ the set of all variable exponents $p \in \mathcal{B}_M(\mathbb{R})$ for which there exist constants $p_0 \in (1, \infty)$, $\theta \in (0,1)$, and a variable exponent $p_1 \in \mathcal{B}_M(\mathbb{R})$ such that

$$\frac{1}{p(x)} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1(x)}, \quad x \in \mathbb{R}.$$

- (Rabinovich, Samko, 2008)
$$LH(\mathbb{R}) \subset \mathcal{B}_M^*(\mathbb{R})$$
- (trivial)
$$\mathcal{B}_M^*(\mathbb{R}) \subset \mathcal{B}_M(\mathbb{R})$$
- (A.K., Spitkovsky, August 2011, unpublished)
$LH(\mathbb{R})$ is a proper subset of $\mathcal{B}_M^*(\mathbb{R})$
- (Diening, October 2011)
$$\mathcal{B}_M^*(\mathbb{R}) = \mathcal{B}_M(\mathbb{R}).$$

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