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The maximal operator in generalized Orlicz spaces

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Generalized Orlicz spaces

Lebesgue -> Orlicz -> generalized Orlicz

$$\int |f|^p \, dx \quad \text{to} \quad \int \varphi(|f|) \, dx \quad \text{to} \quad \int \varphi(x, |f|) \, dx.$$

Or Lebesgue -> variable exponent -> generalized Orlicz
Studied since the 1940's; monograph by Musielak (1983)

Examples

Lebesgue space $\int \varphi(|f|)\,dx$

Exponential space $\int \exp(|f|) - 1\,dx$

Variable exponent space $\int |f|^{p(x)}\,dx$

log-type space $\int |f|^{p(x)} \log(e + |f|)^{p(x)}\,dx$

(p, q)-type space $\int |f|^p + a(x)|f|^q\,dx$

Example continued

log-type space $\int |f|^{p(x)} \log(e+|f|)^{p(x)}\,dx$ studied e.g. in

- Mizuta, Ohno, Shimomura, JMAA 2008
- Hästö, Mizuta, Ohno, Shimomura, Glasgow MJ 2010
- Maeda, Mizuta, Ohno, Shimomure, AASF 2010
- Harjulehto, Hästö, Mizuta, Shimomura, Manus Math 2011
- Mizuta, Shimomura, JMAA 2012
- Mizuta, Nakai, Ohno, Shimomura, Rev Mat Complutense 2012
- Maeda, Mizuta, Shimomura, Nonlinear Anal 2015

Example continued

(p, q)-type space $\int |f|^p + a(x)|f|^q \, dx$ studied in

- Baroni, Colombo, Mingione, Nonlinear Anal 2015
- Baroni, Colombo, Mingione, St Petersburg MJ
- Colombo, Mingione, ARMA
- Colombo, Mingione, ARMA 2015

Motivation

- covers both variable exponent and Orlicz
- fluid dynamics models (Wróblewska-Kamińska, Nonlinearity 2014)
- existence of solutions to parabolic equations with generalized Orlicz growth (Świerczewska-Gwiazda, Nonlinear Anal 2014)
- regularity of minimizers of energies

$$\int |\nabla u|^{p(x)} \log(e+|\nabla u|)\,dx \quad \text{and} \quad \int |\nabla u|^p + a(x)\,|\nabla u|^q\,dx,$$

by Giannetti and Passarelli di Napoli (JDE 2013) and Colombo and Mingione (ARMA 2015), respectively.

Assumptions

(A0) There exists $\beta \in (0,1)$ such that $1 \leqslant \varphi(x, \frac{1}{\beta})$ and $\varphi(x, \beta) \leqslant 1$,

(A1) There exists $\beta \in (0,1)$ such that $\beta\varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t)$ for every $t \in \left[1, \frac{1}{|B|}\right]$, every $x, y \in B$ and every ball B with $|B| \leqslant 1$,

(A2) $L^{\varphi(\cdot)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) = L^{\varphi_\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $\varphi_\infty(t) := \limsup_{|x|\to\infty} \varphi(x,t)$.

The maximal operator

Lemma

Let $\varphi \in \Phi$ and $\beta > 1$ be such that $s \mapsto s^{-\beta}\varphi(s)$ is increasing. Then for each $\gamma \in (1, \beta)$ there exists $\psi \in \Phi$ equivalent to φ such that $\psi^{1/\gamma}$ is convex.

Theorem

Let $\varphi \in \Phi(\mathbb{R}^n)$ satisfy assumptions (A0)–(A2) and suppose that $\gamma > 1$ is such that $s \mapsto s^{-\gamma}\varphi(x, s)$ is increasing for every $x \in \mathbb{R}^n$. Then

$$M : L^{\varphi(\cdot)}(\mathbb{R}^n) \to L^{\varphi(\cdot)}(\mathbb{R}^n)$$

is bounded.

Remarks

The boundedness of the maximal operator in generalized Orlicz spaces was first shown by Maeda, Mizuta, Ohno and Shimomura (2013) in the doubling case with different methods.

These authors have also studied other operators in generalized Orlicz spaces since then.

If $\varphi(x,t) = t^p + a(x)t^q$, then conditions (A0)–(A2) are equivalent to the fact that $\frac{q}{p} \leqslant 1 + \frac{\alpha}{n}$ where $a \in C^\alpha$. This is exactly the same condition found by Colombo and Mingione.

Proof.

Let $\psi \in \Phi$ be as in the lemma. It suffices to show that $M : L^\psi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$. Since $\psi^{1/\gamma}$ is convex, it follows from Jensen's inequality that

$$\psi(\epsilon Mf) = \big(\psi^{\frac{1}{\gamma}}(\epsilon Mf)\big)^\gamma \leqslant \big(M\big(\psi^{\frac{1}{\gamma}}(\epsilon f)\big)\big)^\gamma.$$

Let $f \in L^\psi(\mathbb{R}^n)$ and $\epsilon := \|f\|_\psi^{-1}$ so that $\varrho_\psi(\epsilon f) \leqslant 1$. Since M is bounded in $L^\gamma(\mathbb{R}^n)$, we obtain that

$$\int_{\mathbb{R}^n} \big(M\big(\psi^{\frac{1}{\gamma}}(\epsilon f)\big)\big)^\gamma \, dx \lesssim \int_{\mathbb{R}^n} \big(\psi^{\frac{1}{\gamma}}(\epsilon f)\big)^\gamma \, dx = \int_{\mathbb{R}^n} \psi(\epsilon f) \, dx \leqslant 1.$$

Hence $\varrho_\psi(\epsilon Mf) \lesssim 1$, which implies that $\|\epsilon Mf\|_\psi \lesssim 1$. Dividing by ϵ, we find that $\|Mf\|_\psi \lesssim \frac{1}{\epsilon} = \|f\|_\psi$, which completes the proof. □

Auxiliary results

Shimomura et al.'s results are included, on account of the following lemma.

Lemma

Suppose that $\varphi : [0, \infty) \to [0, \infty)$ *is doubling and that* $s \mapsto \frac{\varphi(s)}{s}$ *is increasing. Then* φ *is equivalent to a convex function* $\psi \in \Phi$.

Every Φ-function satisfying (A0)–(A2) is equivalent to a normalized Φ-function.

Definition

We say that $\varphi \in \Phi_1(\mathbb{R}^n)$ is a *normalized* Φ*-function* if $\varphi(x, t) = \varphi_\infty(t)$ for $t \in [0, 1]$ and there exists $\beta > 0$ such that

$$\beta \varphi^{-1}(x, t) \leqslant \varphi^{-1}(y, t)$$

for every $t \in \left[0, \frac{1}{|B|}\right]$, every $x, y \in B$ and every ball B.

References

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- F.-Y. Maeda, Y. Mizuta, T. Ohno and T. Shimomura: Boundedness of maximal operators and Sobolev's inequality on Musielak–Orlicz–Morrey spaces. Bull. Sci. Math. 137 (2013), 76–96.
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