Several Kinds of Variable Exponent Spaces

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The theory of differential forms is an approach to multi-variable calculus that is independent of coordinates and provides a better definition for integrals. Differential forms have played an important role in physical laws of thermodynamics, analytical mechanics, and physical theories, in particular Maxwell's theory, and the Yang-Mills theory, the theory of relativity.

Let (M, g) be an n-dimensional smooth orientable Riemannian manifold and $\mathrm{d}\mu = \sqrt{\det(g_{ij})}\mathrm{d}x$ be the Riemannian volume element on (M, g), where g_{ij} are the components of the Riemannian metric g in the chart (U_α, f_α) and $\mathrm{d}x$ is the Lebesgue volume element of $\mathbb{R}^n$.
Let $x^1, \ldots, x^n$ be the orientable coordinates on M. Each differential k-form u can be written as

$$u = \sum_{1 \le i_1 < \ldots < i_k \le n} u_{i_1,\ldots,i_k} \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} = \sum_{I \in \Lambda(k,n)} u_I \mathrm{d}x^I,$$

where $\Lambda(n-k, n)$ is the set of ordered multi-indices $I = (i_1, i_2, \ldots, i_k)$ of integers $1 \le i_1 < i_2 < \ldots < i_k \le n$.
Let $\Lambda^k M$ be the set of differential k-forms on M, then the Grassman algebra $\Lambda M = \bigoplus_{k=0}^{n} \Lambda^k M$ is a graded algebra.

Let $\mathcal{P}(M)$ be the set of all measurable functions $p : M \to [1, \infty]$. For $p(m) \in \mathcal{P}(M)$, put

$$M_1 = M_1^p = \{m \in M : p(m) = 1\},$$

$$M_\infty = M_\infty^p = \{m \in M : p(m) = \infty\},$$

$$M_0 = M \backslash (M_1 \cup M_\infty),$$

$$p_* = \inf_{M_0} p(m),$$

$$p^* = \sup_{M_0} p(m),$$

$$\mathcal{P}_1(M) = \mathcal{P}(M) \cap L^\infty(M),$$

$$\mathcal{P}_2(M) = \{p \in \mathcal{P}_1(M) : \inf_M p(m) > 1\}.$$

For a differential k-form u on M, we define the functional

$$\rho_{p(m),\Lambda^k M}(u) = \int_{M\setminus M_\infty} |u|^{p(m)} \mathrm{d}\mu + \sup_{M_\infty} |u|.$$

The Lebesgue space $L^{p(m)}(\Lambda^k M)$ is the space of differential forms u such that

$$\rho_{p(m),\Lambda^k M}(\lambda u) < \infty \text{ for some } \lambda = \lambda(u) > 0$$

with the following norm

$$\|u\|_{L^{p(m)}(\Lambda^k M)} = \inf\left\{\lambda > 0 : \rho_{p(m),\Lambda^k M}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

The exterior Sobolev space $W^{1,p(m)}(\Lambda^k M)$ consists of such forms $u \in L^{p(m)}(\Lambda^k M)$ for which $\mathrm{d}u \in L^{p(m)}(\Lambda^{k+1} M)$. Here $\mathrm{d}u$ is the weak exterior differential of u. The norm is defined by

$$\|u\|_{W^{1,p(m)}(\Lambda^k M)} = ||u||_{L^{p(m)}(\Lambda^k M)} + \|\mathrm{d}u\|_{L^{p(m)}(\Lambda^{k+1} M)}.$$

The space $W_0^{1,p(m)}(\Lambda^k M)$ is defined as the closure of $C_c^\infty(\Lambda^k M)$ in $W^{1,p(m)}(\Lambda^k M)$.

Remark

(1) $L^{p(m)}(\Lambda^0 M)$, $W^{1,p(m)}(\Lambda^0 M)$ and $W_0^{1,p(m)}(\Lambda^0 M)$ are spaces of functions on M and we denote them by $L^{p(m)}(M)$, $W^{1,p(m)}(M)$ and $W_0^{1,p(m)}(M)$ respectively. For these spaces, see also Gaczkowski, Michal; Gorka, Przemyslaw. Sobolev spaces with variable exponents on Riemannian manifolds. NONLINEAR ANALYSIS-TMA, vol. 92, 47-59, 2013.
(2) Let M be a bounded domain $\Omega \subset \mathbb{R}^n$, then we get variable exponent spaces of differential forms in the setting of Euclidean spaces, i. e. $L^{p(m)}(\Omega, \Lambda^k)$, $W^{1,p(m)}(\Omega, \Lambda^k)$ and $W_0^{1,p(m)}(\Omega, \Lambda^k)$. Furthermore $L^{p(m)}(\Omega, \Lambda^0)$, $W^{1,p(m)}(\Omega, \Lambda^0)$ and $W_0^{1,p(m)}(\Omega, \Lambda^0)$ become $L^{p(m)}(\Omega)$, $W^{1,p(m)}(\Omega)$ and $W_0^{1,p(m)}(\Omega)$ respectively.

Theorem

If $p(m) \in \mathcal{P}(M)$*, then the inequality*

$$\int_M |\langle u, v\rangle| d\mu \leq r_p ||u||_{L^{p(m)}(\Lambda^k M)} ||v||_{L^{p'(m)}(\Lambda^k M)}$$

holds for every $u \in L^{p(m)}(\Lambda^k M), \;\; v \in L^{p'(m)}(\Lambda^k M)$.

Theorem

If $p(m) \in \mathcal{P}_1(M)$*, then* $\lim_{t\to\infty} \rho_{p(m),\Lambda^k M}(u_t) = 0$ *if and only if* $\lim_{t\to\infty} ||u_t||_{L^{p(m)}(\Lambda^k M)} = 0$.

Theorem

If $p(m) \in \mathcal{P}_1(M)$ *and* $\mu(M) < \infty$, *then*

$$\lim_{t \to \infty} ||u_t - u||_{L^{p(m)}(\Lambda^k M)} = 0$$

if and only if u_t *converges to* u *on* M *in measure and*

$$\lim_{t \to \infty} \rho_{p(m),\Lambda^k M}(u_t) = \rho_{p(m),\Lambda^k M}(u).$$

Theorem

If $p(m) \in \mathcal{P}_1(M)$, *then* $u \in L^{p(m)}(\Lambda^k M)$ *is absolutely continuous with respect to the norm* $|| \cdot ||_{L^{p(m)}(\Lambda^k M)}$, *i.e.* $\lim_{\mu(G) \to 0} ||u\chi_G||_{L^{p(m)}(\Lambda^k M)} = 0$.

Theorem

If $p(m) \in \mathcal{P}_1(M)$, then $C_c^\infty(\Lambda^k M)$ is dense in $L^{p(m)}(\Lambda^k M)$ and further $L^{p(m)}(\Lambda^k M)$ is separable.

Theorem

If $p(m) \in \mathcal{P}(M)$, then the space $L^{p(m)}(\Lambda^k M)$ is complete.

Theorem

If $p(m) \in \mathcal{P}_2(M)$, then the space $L^{p(m)}(\Lambda^k M)$ is reflexive.

Theorem

If $p(m) \in \mathcal{P}_2(M)$, then the exterior Sobolev space $W^{1,p(m)}(\Lambda^k M)$ is a separable, reflexive Banach space.

Theorem

Let $0 < \mu(M) < \infty$. If $p(m), q(m) \in \mathcal{P}(M)$ and $p(m) \leq q(m)$ a.e. $m \in M$, then $L^{q(m)}(\Lambda^k M) \curvearrowright L^{p(m)}(\Lambda^k M)$ and the norm of the embedding operator does not exceed $\mu(M) + 1$.

Theorem

Let M be a compact Riemannian manifold with a smooth boundary or without boundary and $p(m), q(m) \in C(\overline{M}) \cap \mathcal{P}_1(M)$. Assume that $p(m) < n$, $q(m) < \frac{np(m)}{n-p(m)}$, $m \in \overline{M}$. Then $W^{1,p(m)}(M) \curvearrowright L^{q(m)}(M)$ is a continuous and compact embedding.

Next we show an application of exterior Sobolev spaces on Riemannian manifold. We assume that $\Omega \subset M$ is a bounded domain with smooth boundary and $p(m) \in \mathcal{P}_2(\Omega)$. The nonhomogeneous $p(m)$-harmonic equation for differential forms with variable growth on Ω is

$$\begin{cases} \mathrm{d}^\star(\mathrm{d}u|\mathrm{d}u|^{p(m)-2}) + u|u|^{p(m)-2} = f(m), & m \in \Omega \\ u(m) = 0, & m \in \partial\Omega \end{cases}$$

where the codifferential operator $\mathrm{d}^\star$ is defined by the Hodge star operator $\star$ and the exterior differentiation d as

$$\mathrm{d}^\star u = (-1)^{n(k+1)+1} \star \mathrm{d} \star u.$$

Theorem

If $f(m) \in [W_0^{1,p(m)}(\Lambda^{k-1}\Omega)]'$, then the Dirichlet problem above has a unique weak solution in $W_0^{1,p(m)}(\Lambda^{k-1}\Omega)$.

Corollary

If $f(m) \in [W_0^{1,p(m)}(\Omega)]'$, then the Dirichlet problem

$$\begin{cases} -div(\nabla u|\nabla u|^{p(m)-2}) + u|u|^{p(m)-2} = f(m), & m \in \Omega \\ u(m) = 0, & m \in \partial\Omega \end{cases}$$

has a unique weak solution in $W_0^{1,p(m)}(\Omega)$.

Clifford algebras were introduced by Clifford as geometric algebras in 1878, which are a generalization of the complex numbers, the quaternions, and the exterior algebras. As an active branch of mathematics over the past 40 years, Clifford analysis usually studies the solutions of the Dirac equations for functions defined on domains in Euclidean space and taking value in Clifford algebras.

Let $\mathrm{C}\ell_n$ denote the real universal Clifford algebras over $\mathbb{R}^n$, then

$$\mathrm{C}\ell_n = \mathrm{span}\{e_0, e_1, e_2, \ldots, e_n, e_1 e_2, \ldots, e_{n-1} e_n, \ldots, e_1 e_2 \ldots e_n\}$$

where $e_0 = 1$ (the identity element in $\mathbb{R}^n$), $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_i e_j + e_j e_i = -2\delta_{ij}$. For $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ with $1 \leq i_1 < i_2 < \cdots < i_n \leq n$, put $e_I = e_{i_1} e_{i_2} \ldots e_{i_r}$, while for $I = \emptyset$, $e_\emptyset = e_0$. For $0 \leq r \leq n$ fixed, the space Cl_n^r is defined by

$$\mathrm{C}\ell_n^r = \mathrm{span}\{e_I : \mathrm{card}(I) = r\}.$$

The Clifford algebras $\mathrm{C}\ell_n$ is a graded algebra as

$$\mathrm{C}\ell_n = \bigoplus_{r=0}^{n} \mathrm{C}\ell_n^r.$$

It is customary to identify $\mathbb{R}$ with $\mathrm{C}\ell_n^0$ and identify $\mathbb{R}^n$ with $\mathrm{C}\ell_n^1$ respectively. For $u \in \mathrm{C}\ell_n$, $[u]_0$ denotes the scalar part of u, that is the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{(e_{i_1} e_{i_2} \ldots e_{i_r})} = (-1)^{\frac{r(r+1)}{2}} e_{i_1} e_{i_2} \ldots e_{i_r}.$$

We denote $(A, B) = [\overline{A}B]_0$. Then an inner product is thus obtained, leading to the norm $|\cdot|$ on $\mathrm{C}\ell_n$ given by

$$|A|^2 = [\overline{A}A]_0.$$

A Clifford-valued function $u : \Omega \to C\ell_n$ can be written as $u = \Sigma_I u_I e_I$, where the coefficients $u_I : \Omega \to \mathbb{R}$ are real valued functions.
The Dirac operator on Euclidean space used here is as follows:

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} e_j \partial_j.$$

If u is C^1 real-valued function defined on a domain Ω in $\mathbb{R}^n$, then $Du = \nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_n u)$, where ∇ is the distributional gradient. Further $D^2 = -\Delta$, where Δ is the Laplace operator which operates only on coefficients.

Let $p(x) \in \mathcal{P}(\Omega)$. We define the Clifford valued variable exponent Lebesgue space by

$$L^{p(x)}(\Omega, \mathrm{C}\ell_n) = \{u \in \mathrm{C}\ell_n : u = \sum_I u_I e_I, u_I \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} = \|\sum_I u_I e_I\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} = \sum_I \|u_I\|_{L^{p(x)}(\Omega)}$$

and the Clifford valued variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega, \mathrm{C}\ell_n) = \{u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n) : \nabla u \in (L^{p(x)}(\Omega, \mathrm{C}\ell_n))^n\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} = \|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} + \|\nabla u\|_{(L^{p(x)}(\Omega,\mathrm{C}\ell_n))^n}.$$

By $C^\infty(\Omega, \mathrm{C}\ell_n)$ denote the space of Clifford-valued functions in Ω whose coefficients are infinitely differentiable in Ω and by $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ denote the subspace of $C^\infty(\Omega, \mathrm{C}\ell_n)$ with compact support in Ω. Denote $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ by the closure of $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ in $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

Remark

$u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ can be understood coordinatewise. For example, $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ means that $\{u_I\} \subset L^{p(x)}(\Omega)$ for $u = \sum_I u_I e_I \in \mathrm{C}\ell_n$ with the norm $\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} = \sum_I \|u_I\|_{L^{p(x)}(\Omega)}$. A simple computation shows that $\|u\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}$ is equivalent to $\||u|\|_{L^{p(x)}(\Omega)}$. In the same way, the spaces $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ can be understood similarly.

Theorem

If $p(x) \in \mathcal{P}(\Omega)$, then the inequality

$$\int_M |uv|dx \le r_p ||u||_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} ||v||_{L^{p'(x)}(\Omega, \mathrm{C}\ell_n)}$$

holds for every $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, $v \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$.

Theorem

If $p(x) \in \mathcal{P}_1(\Omega)$, then $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ is dense in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ and further $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is separable.

Theorem

If $p(x) \in \mathcal{P}(\Omega)$, then the space $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is complete.

Theorem

If $p(x) \in \mathcal{P}_2(\Omega)$, then the space $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is reflexive.

Theorem

If $p(x) \in \mathcal{P}_2(\Omega)$, then the Clifford valued variable exponent Sobolev space $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is a separable, reflexive Banach space.

The Teodorescu operator is defined by $Tu(x) = \int_\Omega G(x-y)u(y)\mathrm{d}y$, where $G(x)$ is the generalized Cauchy kernel defined by $G(x) = -\dfrac{1}{\omega_n}\dfrac{x}{|x|^n}$ and ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$.

In the rest part of this section, let $p \in \mathcal{P}_2(\Omega)$ be log-Hörlder continuous. Then the operator T can be uniquely extended to a bounded linear operator $\widetilde{T} : W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n) \to L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ where $W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is the dual of $W_0^{1,p'(x)}(\Omega, \mathrm{C}\ell_n)$ and the Dirac operator D can be uniquely extended to a bounded linear operator $\widetilde{D} = -\Delta T : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

It is easy to get that the Dirichlet problem of Poission equation with homogeneous boundary data $-\Delta u = f,\ x \in \Omega$, has a unique weak solution $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ for each $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$. We denote by Δ_0^{-1} the solution operator.

The variable exponent Lebesgue space admits the direct decomposition:

$$L^{p(x)}(\Omega, \mathrm{C}\ell_n) = (\ker \widetilde{D} \cap L^{p(x)}(\Omega, \mathrm{C}\ell_n)) \oplus DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$$

with respect to the Clifford-valued product. From this decomposition we can get the following projections:

$$P : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to \ker \widetilde{D} \cap L^{p(x)}(\Omega, \mathrm{C}\ell_n),$$

$$Q : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n),$$

where $Q = D\Delta_0^{-1}\widetilde{D}, \; P = I - Q$.

Consider the Stokes system which consists in finding a solution (u, π):

$$\begin{aligned} &\widetilde{D}Du + \frac{1}{\mu}\widetilde{D}\pi = \frac{\rho}{\mu}f, \qquad x \in \Omega, \\ &[Du]_0 = 0, \quad x \in \Omega, \\ &u = 0, \qquad\quad x \in \partial\Omega. \end{aligned}$$

Here u is the velocity, π is the hydrostatic pressure, ρ is the density, μ is the viscosity, f is the vector of the external forces and the scalar function $[Du]_0$ is a measure of the compressibility of fluid.

Theorem

Suppose $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$. *Then the Stokes system has a unique solution* $(u, \pi) \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega)$ *in the form:*

$$u + \frac{1}{\mu} TQ\pi = \frac{\rho}{\mu} TQ\widetilde{T} f$$

with the estimate

$$\|u\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} + \|Q\pi\|_{L^{p(x)}(\Omega)} \leq C\|f\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

Here $C > 1$ *is a constant and* π *is unique up to a constant.*

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space and D be a bounded open subset of $\mathbf{R}^n$ $(n > 1)$. Let λ be a product measure on $D \times \Omega$ and $u(x, \omega)$ be a measurable stochastic field on $D \times \Omega$. Let $p(x, \omega) \in \mathcal{P}(D \times \Omega)$. On the set of all stochastic fields defined on $D \times \Omega$, the functional ρ_p is defined by

$$\rho_p(u) = E\left(\int_{D_0} |u(x,\omega)|^{p(x,\omega)} dx + \sup_{D_\infty} |u(x,\omega)|\right).$$

Definition

The space $L^{p(x,\omega)}(D \times \Omega)$ is the set of measurable stochastic fields u on $D \times \Omega$ such that $\rho_p(ku) < \infty$ for some $k = k(u) > 0$ and it is endowed with the following norm:

$$\|u\|_p = \inf\{k > 0 : \rho_p(\frac{u}{k}) \leq 1\}.$$

Definition

The space $W^{k,p(x,\omega)}(D \times \Omega)$ is the set of stochastic fields such that $D^\alpha u \in L^{p(x,\omega)}$, $|\alpha| \leq k$ and it is endowed with the following norm:

$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_p.$$

Definition

The space $W_0^{k,p(x,\omega)}(D \times \Omega)$ is the closure of $C(D \times \Omega) = \{u : u(\cdot, \omega) \in C_0^\infty(D) \text{ for each } \omega \in \Omega\}$ in $W^{k,p(x,\omega)}(D \times \Omega)$.

Theorem

If $p(x,\omega) \in \mathcal{P}(D \times \Omega)$, then the inequality

$$E\left(\int_D |f(x,\omega)g(x,\omega)|dx\right) \le C\|f\|_p\|g\|_{p'}$$

holds for every $f \in L^{p(x,\omega)}(D \times \Omega)$ and $g \in L^{p'(x,\omega)}(D \times \Omega)$ with the constant C dependent on $p(x,\omega)$ only.

Theorem

If $p(x,\omega) \in \mathcal{P}(D \times \Omega)$, then the space $L^{p(x,\omega)}(D \times \Omega)$ is complete.

Theorem

If $p(x,\omega) \in \mathcal{P}_2(D \times \Omega)$, then then the space $L^{p(x,\omega)}(D \times \Omega)$ is reflexive.

Theorem

If $p(x,\omega) \in \mathcal{P}_2(D \times \Omega)$, then the space $W^{k,p(x,\omega)}(D \times \Omega)$ is a reflexive Banach space.

Theorem

Let $p, q \in \mathcal{P}(D \times \Omega)$ *and* $1 < p^- \leq p(x,\omega) \leq p^+ < N$. *If*

(1) $p(x,\omega) \leq q(x,\omega) \leq p^*(x,\omega) = \frac{Np(x,\omega)}{N-p(x,\omega)}$;

(2) *there exits a constant* L *such that* $|q(x_1,\omega) - q(x_2,\omega)| \leq L|x_1 - x_2|$;

(3) $\inf_{(x,\omega)\in D\times\Omega} \left(\frac{N-1}{N} q(x,\omega) - p(x,\omega)\right) > 0$;

then there is a continuous embedding from $W^{1,p(x,\omega)}(D \times \Omega)$ *to* $L^{q(x,\omega)}(D \times \Omega)$.

Theorem

If $p : \overline{D} \times \Omega \to \mathbf{R}$ *satisfies*

$$|p(x_1, \omega) - p(x_2, \omega)| \leq L|x_1 - x_2|$$

for some constant L*,* $q : \overline{D} \times \Omega \to \mathbf{R}$ *is measurable and satisfies*

$$p(x, \omega) \leq q(x, \omega) \leq p^*(x, \omega) = \frac{Np(x, \omega)}{N - kp(x, \omega)} \text{ a.e.,}$$

then there is a continuous embedding

$$W^{k,p(x,\omega)}(D \times \Omega) \to L^{q(x,\omega)}(D \times \Omega).$$

Next we give an application to the stochastic partial differential equations with stochastic field growth. We will discuss the existence and uniqueness of weak solution for the following equation:

$$\begin{cases} -\mathrm{div} A(x,\omega,u,\nabla u) + B(x,\omega,u,\nabla u) = f(x,\omega), & (x,\omega)\in D\times\Omega, \\ u = 0 & (x,\omega)\in \partial D\times\Omega. \end{cases}$$

$A(x,\omega,s,\xi)$ and $B(x,\omega,s,\xi)$ are Carathéodory functions, which are measurable stochastic fields on $D\times\Omega$ and continuous for s and ξ.

$A : \mathbf{R}^n \times \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n,\ B : \mathbf{R}^n \times \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ satisfy the following growth conditions:

$$(H1)|A(x,\omega,s,\xi)| \le \beta_1(x,\omega)|\xi|^{p(x,\omega)-1} + \beta_2(x,\omega)|s|^{p(x,\omega)-1} + K_1(x,\omega),$$
$$|B(x,\omega,s,\xi) \le \beta_3(x,\omega)|\xi|^{p(x,\omega)-1} + \beta_4(\omega)|s|^{p(x,\omega)-1} + K_2(x,\omega);$$
$$(H2)E((A(x,\omega,s_1,\xi) - A(x,\omega,s_2,\eta))(\xi-\eta) + (B(x,\omega,s_1,\xi)$$
$$- B(x,\omega,s_2,\eta))(s_1 - s_2)) > 0, \xi \ne \eta \text{ or } s_1 \ne s_2;$$
$$(H3)E(A(x,\omega,s,\xi)\xi + B(x,\omega,s,\xi)s) \ge \beta E(|\xi|^{p(x,\omega)} + |s|^{p(x,\omega)}) \text{ a.e.};$$

where $K_1(x,\omega), K_2(x,\omega) \in L^{p'(x,\omega)}(D \times \Omega)$, $\beta > 0$, $\beta_i(x,\omega)$ $(i = 1, \cdots, 4)$ are nonnegative bounded measurable stochastic fields.

Theorem

Under conditions (H1)-(H3), there exists a unique weak solution $u \in W_0^{1,p(x,\omega)}(D \times \Omega)$ to the equation for any $f \in L^{p'(x,\omega)}(D \times \Omega)$.

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Thanks!