Approximation numbers of a Sobolev embedding

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- Banach spaces X, Y; $T \in B(X, Y)$.

$$a_m(T) := \inf \{\|T - F\| : F \in B(X, Y),\ \text{rank } F < m\}$$

is the m^{th} approximation number of T.

- $\{a_m(T)\}_{m\in\mathbb{N}}$ is non-increasing and bounded below: $\alpha(T) = \lim_{m\to\infty} a_m(T)$.
- $\alpha(T) = 0 \Rightarrow T$ compact; converse true if Y has approximation property.
- Connection with eigenvalues: when X, Y are Hilbert spaces and T is compact,

$$a_m(T) = \lambda_m(|T|),$$

the m^{th} eigenvalue of $|T| := (T^*T)^{1/2}$.

- Outside Hilbert spaces, if $T \in K(X)$,

$$|\lambda_m(T)| = \lim_{k\to\infty} \left\{a_m\left(T^k\right)\right\}^{1/k}.$$

- Rate of decay of $a_m(T)$ ($T \in K(X, Y)$) indicates 'how compact' T is.

- Embeddings of classical Sobolev spaces
- Ω : bounded open subset of $\mathbb{R}^n$; $p \in (1, \infty)$
- $\overset{0}{W}{}^1_p(\Omega)$: completion of smooth functions with compact support in Ω with respect to

$$\left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}.$$

- id: $\overset{0}{W}{}^1_p(\Omega) \to L_p(\Omega)$ natural embedding
- Known that (Birman-Solomyak, for example)

$$a_m(id) \approx m^{-1/n}$$

in the sense that

$$c_1 m^{-1/n} \leq a_m(id) \leq c_2 m^{-1/n}.$$

- If $n = 1$ and $\Omega = (a, b)$, then (Lang-E.)

$$a_m(id) = \gamma_p(b-a)/m, \ \gamma_p = (p')^{1/p} p^{1/p'} (2\pi)^{-1} \sin(\pi/p).$$

Main purpose of talk: to describe extensions of such estimates to spaces with variable exponent.

- Work with Jan Lang and Ales Nekvinda
- $\mathcal{M}(\Omega)$: all measurable, extended real-valued functions on Ω
- $\mathcal{P}(\Omega) \subset \mathcal{M}(\Omega)$: all $p : \Omega \to [1, \infty)$.

$$\rho_p(f) = \int_\Omega |f(x)|^{p(x)}\, dx$$

- $L_p(\Omega) :=$ all $f \in \mathcal{M}(\Omega)$ such that $\rho_p(f/\lambda) < \infty$ for some $\lambda > 0$, endowed with norm

$$\|f\|_p := \inf \{\lambda > 0 : \rho_p(f/\lambda) \le 1\}.$$

- Let $\varepsilon \in (0,1)$ and suppose that $p, q \in \mathcal{P}(\Omega)$ satisfy

$$1 < p(x) \leq q(x) \leq p(x) + \varepsilon \text{ for all } x \in \Omega.$$

  Known that $L_q(\Omega) \hookrightarrow L_p(\Omega)$; let id be the embedding map.
- Desirable to have upper and lower estimates for $\|\mathrm{id}\|$.
- Upper estimate: if $\varepsilon \leq 1/2$, then

$$\|\mathrm{id}\| \leq 1 + K|\Omega|\varepsilon, \quad K = \sup_{0<\alpha\leq 1} \alpha^{1/2}|\log \alpha|.$$

- Lower estimate: when $|\Omega| \geq 1$, let $\varepsilon_0 = 1, L = 0$; when $|\Omega| < 1$, let $\varepsilon_0 = 1/\log(1/|\Omega|), L = \log(1/|\Omega|)$. Then

$$\|\mathrm{id}\| \geq 1 - \varepsilon L \text{ if } 0 \leq \varepsilon < \varepsilon_0.$$

- Sobolev embedding. Let $\mathcal{P}_l(\Omega) \subset \mathcal{P}(\Omega)$ consist of all those bounded p with

$$|p(x) - p(y)| \log \frac{1}{|x-y|} \leq C \text{ whenever } 0 < |x-y| \leq 1/2.$$

- Let $\mathcal{P}_l(\Omega)$; define

$$W_p^1(\Omega) = \{u \in L_p(\Omega) : |\nabla u| \in L_p(\Omega)\};$$

with norm

$$u \mapsto \|u\|_{1,p} := \|u\|_p + \||\nabla u|\|_p$$

it is a Banach space. Closure of $C_0^\infty(\Omega)$ in $W_p^1(\Omega)$ denoted by $\overset{0}{W}{}_p^1(\Omega)$.

- Norm induced on $\overset{0}{W}{}_p^1(\Omega)$ by $\|\cdot\|_{1,p}$ equivalent to $u \longmapsto \||\nabla u|\|_p$;
- suppose henceforth that this equivalent norm is used and $p \in \mathcal{P}_l(\Omega)$.

- Known that id: $\overset{0}{W}{}_p^1(\Omega) \to L_p(\Omega)$ is compact.
- **Theorem** There are positive constants c_1, c_2 such that for all $m \in \mathbb{N}$,

$$c_1 m^{-1/n} \leq a_m(\text{id}) \leq c_2 m^{-1/n}.$$

- Components of proof:
- Cover $\overline{\Omega}$ by congruent cubes: there exists $A \geq 1$ such that for each $k \in \mathbb{N}$, there is a covering by non-overlapping congruent cubes $\{Q_i\}_{i=1}^m$ with

$$k \leq \sharp \{j : Q_j \subset \Omega\} \leq m \leq Ak.$$

- Extension of p
- Estimates on cubes, using approximations of p and knowledge of norm errors committed by such approximations.
- Lower estimate of Bernstein numbers $b_m(\text{id})$. For $T \in B(X, Y)$,

$$b_m(T) := \sup \inf_{x \in X_m \setminus \{0\}} \|Tx\|_Y / \|x\|_X ,$$

sup over all $m-$dimensional linear subspaces X_m of X.
- Combine these estimates, using $b_m(T) \leq a_m(T)$.

- Same estimates hold for the Bernstein, Gelfand and Kolmogorov numbers of id.
- Sharper results (Lang-E) when $n = 1, \Omega = (a, b)$:

$$\lim_{m\to\infty} m a_m(\mathrm{id}) = \frac{1}{2\pi} \int_a^b \left\{ p'(t) p(t)^{p(t)-1} \right\}^{1/p(t)} \sin(\pi/p(t)) dt.$$

Same for Bernstein, Gelfand and Kolmogorov numbers.