Topics in geometric function theory for variable exponent equations

Tomasz Adamowicz
Institute of Mathematics Polish Academy of Sciences

International meeting of AMS-EMS-SPM in Porto
Recent Advances in Variable Exponent Spaces and Non-linear Problems

Research and visit supported by Grant Sonata 5 by National Science Center, UMO 2013/09/D/ST1/03681

June 10-13, 2015

Presentation is based on joint works with:

- Peter Hästö, Turku and Oulu Universities,
- A. Björn, J. Björn, Linköping University,
- N. L. P. Lundström, Umeå University.

Energy functional

$$\min \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} \, \mathrm{d}x.$$

The Euler-Lagrange equation for the above energy functional is

$p(\cdot)$-harmonic equations

$$\Delta_{p(\cdot)}(u) := \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) \; = \; 0,$$

where $u \in W^{1,p(\cdot)}_{loc}(\Omega, \mathbb{R})$ and function $p : \Omega \to [1, \infty]$ is measurable.

Remarks

(1) For $p \equiv const$ we retrieve the *p-harmonic equation*:

$$\mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

Energy functional

$$\min \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} \, \mathrm{d}x.$$

The Euler-Lagrange equation for the above energy functional is

$p(\cdot)$-harmonic equations

$$\Delta_{p(\cdot)}(u) := \mathrm{div}(|\nabla u|^{p(x)-2} \nabla u) \; = \; 0,$$

where $u \in W^{1,p(\cdot)}_{loc}(\Omega, \mathbb{R})$ and function $p : \Omega \to [1, \infty]$ is measurable.

Remarks

(1) For $p \equiv const$ we retrieve the *p-harmonic equation*:

$$\mathrm{div}(\, |\nabla u|^{p-2} \, \nabla u) = 0.$$

(2) The $p(\cdot)$-harmonic operator is the prototypical equation for more general class of PDEs with non-standard growth:

$$\mathrm{div}\, A(x, u, \nabla u) = B(x, u, \nabla u).$$

Energy functional

$$\min \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} \, \mathrm{d}x.$$

The Euler-Lagrange equation for the above energy functional is

$p(\cdot)$-harmonic equations

$$\Delta_{p(\cdot)}(u) := \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) \; = \; 0,$$

where $u \in W^{1,p(\cdot)}_{loc}(\Omega, \mathbb{R})$ and function $p : \Omega \to [1, \infty]$ is measurable.

Remarks

(1) For $p \equiv const$ we retrieve the *p-harmonic equation*:

$$\mathrm{div}(\,|\nabla u|^{p-2}\,\nabla u) = 0.$$

(2) The $p(\cdot)$-harmonic operator is the prototypical equation for more general class of PDEs with non-standard growth:

$$\mathrm{div}\, A(x, u, \nabla u) = B(x, u, \nabla u).$$

(My) motivations for studying variable exponent PDEs

(1) Generalization of the classical results for p-Laplace type equations with $p = const$.

(2) Applications in image processing (Chen, Levine, Rao), fluid dynamics (Diening, Růžička), electrorheological fluids (Acerbi, Mingione).

(My) motivations for studying variable exponent PDEs

(1) Generalization of the classical results for p-Laplace type equations with $p = const$.

(2) Applications in image processing (Chen, Levine, Rao), fluid dynamics (Diening, Růžička), electrorheological fluids (Acerbi, Mingione).

(3) studies of the geometric properties of variable exponent equations, e.g. the structure of the set of critical points, connections to quasiregular mappings, (boundary) Harnack inequalities, Liouville-type theorems, maximum principles on unbounded domains (Phragmén–Lindelöf theorems).

(My) motivations for studying variable exponent PDEs

(1) Generalization of the classical results for p-Laplace type equations with $p = const$.

(2) Applications in image processing (Chen, Levine, Rao), fluid dynamics (Diening, Růžička), electrorheological fluids (Acerbi, Mingione).

(3) studies of the geometric properties of variable exponent equations, e.g. the structure of the set of critical points, connections to quasiregular mappings, (boundary) Harnack inequalities, Liouville-type theorems, maximum principles on unbounded domains (Phragmén–Lindelöf theorems).

The strong $p(\cdot)$-Laplacian, critical points and the homogeneous Harnack inequality

Joint works with P. Hästö

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Suppose that an orientation preserving mapping $f : \Omega \to \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies

$$\|Df(x)\|^n \leqslant K(x,f)|J(x,f)| \quad \text{a.e. } \Omega.$$

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Suppose that an orientation preserving mapping $f : \Omega \to \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies

$$\|Df(x)\|^n \leqslant K(x,f)|J(x,f)| \quad \text{a.e. } \Omega.$$

- If $1 \leq K(x,f) < \infty$ a.e. then we call map f a *mapping of finite distortion.*

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Suppose that an orientation preserving mapping $f : \Omega \to \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies

$$\|Df(x)\|^n \leqslant K(x,f)|J(x,f)| \quad \text{a.e. } \Omega.$$

- If $1 \leq K(x,f) < \infty$ a.e. then we call map f a *mapping of finite distortion*.
- If $1 \leq K(x,f) \equiv K < \infty$, then we call map f *quasiregular*.

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Suppose that an orientation preserving mapping $f : \Omega \to \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies

$$\|Df(x)\|^n \leqslant K(x,f)|J(x,f)| \quad \text{a.e. } \Omega.$$

- If $1 \leq K(x,f) < \infty$ a.e. then we call map f a *mapping of finite distortion*.
- If $1 \leq K(x,f) \equiv K < \infty$, then we call map f *quasiregular*.
- An homeomorphic quasiregular map is called *quasiconformal*.

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Suppose that an orientation preserving mapping $f : \Omega \to \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies

$$\|Df(x)\|^n \leqslant K(x,f)|J(x,f)| \quad \text{a.e. } \Omega.$$

- If $1 \leq K(x,f) < \infty$ a.e. then we call map f a *mapping of finite distortion*.
- If $1 \leq K(x,f) \equiv K < \infty$, then we call map f *quasiregular*.
- An homeomorphic quasiregular map is called *quasiconformal*.
- 1-quasiregular mappings in the plane are holomorphic functions (in general, conformal).

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Suppose that an orientation preserving mapping $f : \Omega \to \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies

$$\|Df(x)\|^n \leqslant K(x,f)|J(x,f)| \quad \text{a.e. } \Omega.$$

- If $1 \leq K(x,f) < \infty$ a.e. then we call map f a *mapping of finite distortion*.
- If $1 \leq K(x,f) \equiv K < \infty$, then we call map f *quasiregular*.
- An homeomorphic quasiregular map is called *quasiconformal*.
- 1-quasiregular mappings in the plane are holomorphic functions (in general, conformal).
- C^1 mappings with non-zero Jacobian are quasiregular.

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Suppose that an orientation preserving mapping $f : \Omega \to \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ satisfies

$$\|Df(x)\|^n \leqslant K(x,f)|J(x,f)| \quad \text{a.e. } \Omega.$$

- If $1 \leq K(x,f) < \infty$ a.e. then we call map f a *mapping of finite distortion*.
- If $1 \leq K(x,f) \equiv K < \infty$, then we call map f *quasiregular*.
- An homeomorphic quasiregular map is called *quasiconformal*.
- 1-quasiregular mappings in the plane are holomorphic functions (in general, conformal).
- C^1 mappings with non-zero Jacobian are quasiregular.

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Bojarski-Iwaniec,'87 ($p \geq 2$); Manfredi,'88 ($1 < p < \infty$)

If u is a nonconstant p-harmonic function in the planar domain, then its complex gradient

$$f = \frac{1}{2}(u_x - iu_y)$$

is a quasiregular mapping.

In a consequence the set of critical points of u is discrete; that is zeros of ∇u are isolated.

Problem:

Describe the set of critical points for the $p(\cdot)$-harmonic function in the plane.

The importance of p-harmonic functions in geometry of planar mappings comes from their relation to the so-called quasiregular mappings.

Bojarski-Iwaniec,'87 ($p \geq 2$); Manfredi,'88 ($1 < p < \infty$)

If u is a nonconstant p-harmonic function in the planar domain, then its complex gradient

$$f = \frac{1}{2}(u_x - iu_y)$$

is a quasiregular mapping.

In a consequence the set of critical points of u is discrete; that is zeros of ∇u are isolated.

Drawbacks of the $p(\cdot)$-Laplace equation

Examples exist showing that the $p(\cdot)$-Laplace equation (and its known modification) fails to have ∇u as a quasiregular map.

Further drawbacks of $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = 0$:

1. lack of scalability: λu is not necessary $p(\cdot)$-harmonic if u is
2. nonhomogeneous Harnack inequality:

$$\sup_B u \leq C(u)(\inf_B u + |B|)$$

The strong $p(\cdot)$-Laplacian

A.–Hästö (2010)

$$\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p$$

defined for $u \in W^{1,p(\cdot)}_{loc}(\Omega)$.

Let $u \in C^2(\Omega)$ and $p \in C^1(\Omega)$. Then the **logarithmic modification** cancels out with the corresponding term on the left hand side:

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) =$$
$$a\, u_{xx} + 2b\, u_{xy} + c\, u_{yy} + |\nabla u|^{p(x)-2} \log(|\nabla u|)\nabla u \cdot \nabla p$$
$$= |\nabla u|^{p(x)-2} \log(|\nabla u|)\nabla u \cdot \nabla p,$$

$$a = a(p, \nabla u), \quad b = b(p, \nabla u), \quad c = c(p, \nabla u).$$

Strong methods can now be applied (see e.g. Gilbarg-Trudinger book).

Let $u \in C^2(\Omega)$ and $p \in C^1(\Omega)$. Then the **logarithmic modification** cancels out with the corresponding term on the left hand side:

$$
\begin{aligned}
&\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \\
&a\, u_{xx} + 2b\, u_{xy} + c\, u_{yy} + |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p \\
&\qquad\qquad\qquad = |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p,
\end{aligned}
$$

$$
a = a(p, \nabla u), \quad b = b(p, \nabla u), \quad c = c(p, \nabla u).
$$

Strong methods can now be applied (see e.g. Gilbarg-Trudinger book).

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is:

(1) *Lipschitz continuous*

(2) $1 < p^- \le p(x) \le p^+ < \infty$

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is:

(1) *Lipschitz continuous*

(2) $1 < p^- \leq p(x) \leq p^+ < \infty$

Then:

(A) *there exists a weak solution $u \in C^{1,\gamma}(\overline{\Omega})$ of*

$$\begin{cases} \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad (\star)$$

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is:

(1) *Lipschitz continuous*

(2) $1 < p^- \leq p(x) \leq p^+ < \infty$

Then:

(A) *there exists a weak solution $u \in C^{1,\gamma}(\overline{\Omega})$ of*

$$\begin{cases} \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad (\star)$$

(B) *Moreover, the complex gradient $\frac{1}{2}(u_x - iu_y)$ of u is K-quasiregular with*

$$K_p(\cdot) = \frac{1}{2}\left(p(x) - 1 + \frac{1}{p(x)-1}\right) < \infty.$$

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is:

(1) *Lipschitz continuous*

(2) $1 < p^- \leq p(x) \leq p^+ < \infty$

Then:

(A) *there exists a weak solution $u \in C^{1,\gamma}(\overline{\Omega})$ of*

$$\begin{cases} \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{$\star$}$$

(B) *Moreover, the complex gradient $\frac{1}{2}(u_x - iu_y)$ of u is K-quasiregular with*

$$K_p(\cdot) = \frac{1}{2}\left(p(x) - 1 + \frac{1}{p(x)-1}\right) < \infty.$$

In fact, the more general result holds.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is bounded such that

In fact, the more general result holds.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is bounded such that

1. *the set $Y := \{p = 1\}$ has Hausdorff 1-measure equal to zero*

In fact, the more general result holds.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is bounded such that

1. *the set $Y := \{p = 1\}$ has Hausdorff 1-measure equal to zero*
2. *p is locally Lipschitz in $\overline{\Omega} \setminus Y$.*

In fact, the more general result holds.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is bounded such that

1. *the set $Y := \{p = 1\}$ has Hausdorff 1-measure equal to zero*
2. *p is locally Lipschitz in $\overline{\Omega} \setminus Y$.*

Then:
(A) *there exists a weak solution of ($\star$).*

In fact, the more general result holds.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is bounded such that

1. *the set $Y := \{p = 1\}$ has Hausdorff 1-measure equal to zero*
2. *p is locally Lipschitz in $\overline{\Omega} \setminus Y$.*

Then:
(A) *there exists a weak solution of $(\star)$.*

(B) *If*

$$K_p(\cdot) \in \mathrm{Exp}\, L(\Omega),$$

with $K_p(\cdot)$ as before, then the complex gradient $\frac{1}{2}(u_x - iu_y)$ of u is a mapping of finite distortion with unbounded distortion.

In fact, the more general result holds.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded C^2 domain and let $g \in C^{1,\gamma}(\partial\Omega)$. Suppose that p is bounded such that

1. *the set $Y := \{p = 1\}$ has Hausdorff 1-measure equal to zero*
2. *p is locally Lipschitz in $\overline{\Omega} \setminus Y$.*

Then:

(A) *there exists a weak solution of $(\star)$.*

(B) *If*

$$K_p(\cdot) \in \operatorname{Exp} L(\Omega),$$

with $K_p(\cdot)$ as before, then the complex gradient $\frac{1}{2}(u_x - iu_y)$ of u is a mapping of finite distortion with <u>unbounded</u> distortion.

Further advantages of new equation

- Homogeneity (on the contrary to the "prototype" $p(\cdot)$-Laplacian)
- The connection between the strong $p(\cdot)$-Laplacian and Δ_∞-Laplacian:

$$\begin{aligned}\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) - |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p \\ = |\nabla u|^{p(x)-4}\big(|\nabla u|^2\Delta u + (p(\cdot)-2)\Delta_\infty u\big).\end{aligned}$$

See e.g. Juutinen–Lukkari-Parviainen (2011).

Further advantages of new equation

- Homogeneity (on the contrary to the "prototype" $p(\cdot)$-Laplacian)
- The connection between the strong $p(\cdot)$-Laplacian and Δ_∞-Laplacian:

$$\begin{aligned}\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u\cdot\nabla p \\ = |\nabla u|^{p(x)-4}\big(\,|\nabla u|^2\Delta u + (p(\cdot)-2)\Delta_\infty u\,\big).\end{aligned}$$

  See e.g. Juutinen–Lukkari-Parviainen (2011).
- The tug-of-war game, see Pérez-Llanos (2013).

Further advantages of new equation

- Homogeneity (on the contrary to the "prototype" $p(\cdot)$-Laplacian)
- The connection between the strong $p(\cdot)$-Laplacian and Δ_∞-Laplacian:

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |\nabla u|^{p(x)-2}\log(|\nabla u|)\nabla u \cdot \nabla p$$
$$= |\nabla u|^{p(x)-4}\big(\,|\nabla u|^2\Delta u + (p(\cdot)-2)\Delta_\infty u\,\big).$$

  See e.g. Juutinen–Lukkari-Parviainen (2011).
- The tug-of-war game, see Pérez-Llanos (2013).

Further advantages of new equation: Harnack inequality

Strong methods apply resulting in e.g.

Theorem (homogeneous Harnack inequality, 2011)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain; let exponent p satisfy the either

1. $1 < p^- \leq p^+ < n$ *and* $\nabla p \in L^n \log L^n(\Omega)$
   or
2. $1 < p^- \leq p^+ < \infty$ *and* $\nabla p \in L^{q(\cdot)}(\Omega)$, *where* $q \geq \max\{p, n\} + \delta$ *for some* $\delta > 0$.

If u is a positive solution of the equation $(\star)$, then

$$\operatorname*{ess\,sup}_{x \in B} u(x) \leq \mathbf{c}(\mathbf{n}, \mathbf{p}^-, \mathbf{p}^+) \operatorname*{ess\,inf}_{x \in B} u(x),$$

for balls B with $2B \Subset \Omega$.

Further advantages of new equation: Global integrability of supersolutions on Hölder domains

Hölder domain

A *Hölder domain* Ω is a proper subdomain of $\mathbb{R}^n$ in which

$$k_\Omega(x, x_0) \leq c \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} + c$$

for some c, $x_0 \in D$ and every $x \in D$. Here k_Ω denotes the *quasihyperbolic metric*,

$$k_\Omega(x, y) := \inf \int_\gamma \frac{ds(z)}{\operatorname{dist}(z, \partial\Omega)},$$

- John domains are Hölder domains.

Further advantages of new equation: Global integrability of supersolutions on Hölder domains

Hölder domain

A *Hölder domain* Ω is a proper subdomain of $\mathbb{R}^n$ in which

$$k_\Omega(x, x_0) \leq c \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} + c$$

for some c, $x_0 \in D$ and every $x \in D$. Here k_Ω denotes the *quasihyperbolic metric*,

$$k_\Omega(x, y) := \inf \int_\gamma \frac{ds(z)}{\operatorname{dist}(z, \partial\Omega)},$$

- John domains are Hölder domains.
- In the plane a simply connected domain is a Hölder domain if and only if its Riemann mapping is Hölder continuous.

Further advantages of new equation: Global integrability of supersolutions on Hölder domains

Hölder domain

A *Hölder domain* Ω is a proper subdomain of $\mathbb{R}^n$ in which

$$k_\Omega(x, x_0) \le c \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} + c$$

for some c, $x_0 \in D$ and every $x \in D$. Here k_Ω denotes the *quasihyperbolic metric*,

$$k_\Omega(x, y) := \inf \int_\gamma \frac{ds(z)}{\operatorname{dist}(z, \partial\Omega)},$$

- John domains are Hölder domains.
- In the plane a simply connected domain is a Hölder domain if and only if its Riemann mapping is Hölder continuous.

(1) Counterexample by **Armitage** for p-integrability of harmonic functions on balls.
p-Integrability of positive superharmonic functions on balls with $0 < p < \frac{n}{n-1}$.

Global integrability of supersolutions: history

(1) Counterexample by **Armitage** for p-integrability of harmonic functions on balls.
p-Integrability of positive superharmonic functions on balls with $0 < p < \frac{n}{n-1}$.

(2) **Lindqvist** proved global integrability of positive p-supersolutions for Hölder domains in $\mathbb{R}^n$.

Global integrability of supersolutions: history

(1) Counterexample by **Armitage** for p-integrability of harmonic functions on balls.
p-Integrability of positive superharmonic functions on balls with $0 < p < \frac{n}{n-1}$.

(2) **Lindqvist** proved global integrability of positive p-supersolutions for Hölder domains in $\mathbb{R}^n$.

(3) **Maasalo** proved global integrability of positive p-superharmonic functions for Hölder domains on metric spaces.

Global integrability of supersolutions: history

(1) Counterexample by **Armitage** for p-integrability of harmonic functions on balls.
p-Integrability of positive superharmonic functions on balls with $0 < p < \frac{n}{n-1}$.

(2) **Lindqvist** proved global integrability of positive p-supersolutions for Hölder domains in $\mathbb{R}^n$.

(3) **Maasalo** proved global integrability of positive p-superharmonic functions for Hölder domains on metric spaces.

The following theorem corresponds to the results by Lindqvist and Maasalo.

Theorem (global integrability of $p(\cdot)$-supersolutions for the strong $p(\cdot)$-Laplacian)

Let Ω be a Hölder domain in $\mathbb{R}^n$ and let

- $\nabla p \in L^n \log L^n(\Omega)$
- $1 < p^- \leq p^+ < \infty$.

The following theorem corresponds to the results by Lindqvist and Maasalo.

Theorem (global integrability of $p(\cdot)$-supersolutions for the strong $p(\cdot)$-Laplacian)

Let Ω be a Hölder domain in $\mathbb{R}^n$ and let

- $\nabla p \in L^n \log L^n(\Omega)$
- $1 < p^- \leq p^+ < \infty$.

If u is a positive $p(\cdot)$-supersolution of the equation $(\star)$, then there exists $q > 0$, depending only on p^-, p^+ and $\|\nabla p\|_{L^n \log L^n(\Omega)}$, such that

$$\int_\Omega u^q \, dx < \infty.$$

The following theorem corresponds to the results by Lindqvist and Maasalo.

Theorem (global integrability of $p(\cdot)$-supersolutions for the strong $p(\cdot)$-Laplacian)

Let Ω be a Hölder domain in $\mathbb{R}^n$ and let

- $\nabla p \in L^n \log L^n(\Omega)$
- $1 < p^- \leq p^+ < \infty$.

If u is a positive $p(\cdot)$-supersolution of the equation $(\star)$, then there exists $q > 0$, depending only on p^-, p^+ and $\|\nabla p\|_{L^n \log L^n(\Omega)}$, such that

$$\int_\Omega u^q \, dx < \infty.$$

Boundary regularity for $p(\cdot)$-harmonic functions

Joint work with A. Björn and J. Björn, Linköping University

Assumptions

- $\operatorname{div}(p(x)|\nabla u|^{p(x)-2}\nabla u) = 0.$
- $1 < p^- \leq p(x) \leq p^+ < \infty, \quad p(\cdot)$ is log-Hölder

Regular and irregular points

Let $x_0 \in \partial\Omega$. Then x_0 is *regular* if

$$\lim_{\Omega \ni y \to x_0} u(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega)$$

and all $p(\cdot)$-harmonic solutions u to the Dirichlet problem with boundary data f.

We also say that x_0 is *irregular* if it is not regular.

Regular and irregular points

Let $x_0 \in \partial\Omega$. Then x_0 is *regular* if

$$\lim_{\Omega \ni y \to x_0} u(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega)$$

and all $p(\cdot)$-harmonic solutions u to the Dirichlet problem with boundary data f.

We also say that x_0 is *irregular* if it is not regular.

Theorem (The Kellogg property, A.–Björn–Björn)

The set of irregular points in $\partial\Omega$ has zero Sobolev $p(\cdot)$-capacity. Moreover, such a set is an F_σ set.

Regular and irregular points

Let $x_0 \in \partial\Omega$. Then x_0 is *regular* if

$$\lim_{\Omega \ni y \to x_0} u(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega)$$

and all $p(\cdot)$-harmonic solutions u to the Dirichlet problem with boundary data f.

We also say that x_0 is *irregular* if it is not regular.

Theorem (The Kellogg property, A.–Björn–Björn)

The set of irregular points in $\partial\Omega$ has zero Sobolev $p(\cdot)$-capacity. Moreover, such a set is an F_σ set.

Irregular points

A point $x_0 \in \partial\Omega$ is regular if the following two conditions hold:

(a) for all $f \in C(\partial\Omega)$ the limit

$$\lim_{\Omega \ni y \to x_0} u(y) \quad \text{exists;}$$

(b) for all $f \in C(\partial\Omega)$ there is a sequence $\{y_j\}_{j=1}^{\infty}$ such that

$$\Omega \ni y_j \to x_0 \text{ and } u(y_j) \to f(x_0), \quad \text{as } j \to \infty.$$

Semiregular and strongly irregular points

We say that $x_0 \in \partial\Omega$ is *semiregular* if (a) holds but not (b); and *strongly irregular* if (b) holds but not (a).

Irregular points

A point $x_0 \in \partial\Omega$ is regular if the following two conditions hold:

(a) for all $f \in C(\partial\Omega)$ the limit

$$\lim_{\Omega \ni y \to x_0} u(y) \quad \text{exists;}$$

(b) for all $f \in C(\partial\Omega)$ there is a sequence $\{y_j\}_{j=1}^{\infty}$ such that

$$\Omega \ni y_j \to x_0 \text{ and } u(y_j) \to f(x_0), \quad \text{as } j \to \infty.$$

Semiregular and strongly irregular points

We say that $x_0 \in \partial\Omega$ is *semiregular* if (a) holds but not (b); and *strongly irregular* if (b) holds but not (a).

Examples of irregular points

- **Zaremba's punctured ball**
  Let $p = 2$ and
  $$\Omega = B((0,0),1) \setminus \{(0,0)\}.$$
  Then $x_0 = (0,0)$ is a semiregular point.
- **The Lebesgue spine**
  Let $p = 2$ and
  $$E = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : t > 0, |x| < e^{-\frac{1}{t}}\}, \quad \Omega = B((0,0,0),1) \setminus \overline{E}.$$
  Then $x_0 = (0,0,0)$ is a strongly irregular point.

Examples of irregular points

- **Zaremba's punctured ball**
  Let $p = 2$ and
  $$\Omega = B((0,0),1) \setminus \{(0,0)\}.$$
  Then $x_0 = (0,0)$ is a semiregular point.
- **The Lebesgue spine**
  Let $p = 2$ and
  $$E = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : t > 0, |x| < e^{-\frac{1}{t}}\}, \quad \Omega = B((0,0,0),1) \setminus \overline{E}.$$
  Then $x_0 = (0,0,0)$ is a strongly irregular point.
- For any compact set K with zero Sobolev p-capacity we can find a set Ω, such that K consists of semiregular points in $\partial\Omega$. Similar construction can be done for strongly irregular points.

Examples of irregular points

- **Zaremba's punctured ball**
  Let $p = 2$ and
  $$\Omega = B((0,0),1) \setminus \{(0,0)\}.$$
  Then $x_0 = (0,0)$ is a semiregular point.
- **The Lebesgue spine**
  Let $p = 2$ and
  $$E = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : t > 0, |x| < e^{-\frac{1}{t}}\}, \quad \Omega = B((0,0,0),1) \setminus \overline{E}.$$
  Then $x_0 = (0,0,0)$ is a strongly irregular point.
- For any compact set K with zero Sobolev p-capacity we can find a set Ω, such that K consists of semiregular points in $\partial\Omega$. Similar construction can be done for strongly irregular points.

Trichotomy

It turns out that for irregular boundary points *exactly one* of the above two properties (a) or (b) fails. A priori one would assume that it is possible that both fail but this cannot happen.

Theorem (Trichotomy, A.–Björn–Björn)

Let $x_0 \in \partial\Omega$. Then x_0 is either regular, semiregular or strongly irregular.

The proof uses removability of sets with zero Sobolev $p(\cdot)$-capacity and the Kellogg property.

Trichotomy

It turns out that for irregular boundary points *exactly one* of the above two properties (a) or (b) fails. A priori one would assume that it is possible that both fail but this cannot happen.

Theorem (Trichotomy, A.–Björn–Björn)

Let $x_0 \in \partial\Omega$. Then x_0 is either regular, semiregular or strongly irregular.

The proof uses removability of sets with zero Sobolev $p(\cdot)$-capacity and the Kellogg property.

Boundary Harnack inequality for $p(\cdot)$-harmonic equation

Joint works with N. L. P. Lundström

Assumptions

- $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = 0.$
- $1 < p^- \leq p(x) \leq p^+ < \infty, \quad p(\cdot)$ is log-Hölder

Boundary Harnack inequalities

The classical boundary Harnack inequality for two positive harmonic functions u and v in Ω asserts the following: if u and v continuously vanish at every regular point of a set $U \cap \partial\Omega$ and are bounded near every irregular point of $U \cap \partial\Omega$, then

$$\frac{u(x)}{u(y)} \frac{v(y)}{v(x)} \leq A,$$

for all points $x, y \in K \cap \Omega$, where $K \subset U$ is compact. A constant A depends only on Ω, K and U.

Similarly inequalities have been studied for nonlinear PDEs, in the context of manifolds or metric spaces:

- Caffarelli-Fabes-Mortola–Salsa, Jerison-Kenig,
- Bañuelos-Bass-Burdzy,
- Aikawa,
- Lewis-Nyström.

Boundary Harnack inequalities

The classical boundary Harnack inequality for two positive harmonic functions u and v in Ω asserts the following: if u and v continuously vanish at every regular point of a set $U \cap \partial\Omega$ and are bounded near every irregular point of $U \cap \partial\Omega$, then

$$\frac{u(x)}{u(y)} \frac{v(y)}{v(x)} \leq A,$$

for all points $x, y \in K \cap \Omega$, where $K \subset U$ is compact. A constant A depends only on Ω, K and U.

Similarly inequalities have been studied for nonlinear PDEs, in the context of manifolds or metric spaces:

- Caffarelli-Fabes-Mortola–Salsa, Jerison-Kenig,
- Bañuelos-Bass-Burdzy,
- Aikawa,
- Lewis-Nyström.

Theorem (A.-Lundström (2014))

(a) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b.

(b) Let $w \in \partial\Omega$, $0 < r < r_b$.

Boundary Harnack inequality: the $p(\cdot)$-Laplacian

Theorem (A.-Lundström (2014))

(a) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b.

(b) Let $w \in \partial\Omega$, $0 < r < r_b$.

(c) Let p be a bounded Lipschitz continuous variable exponent.

Theorem (A.-Lundström (2014))

(a) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b.

(b) Let $w \in \partial\Omega$, $0 < r < r_b$.

(c) Let p be a bounded Lipschitz continuous variable exponent.

Assume that u and v are positive $p(\cdot)$-harmonic functions in $\Omega \cap B(w, r)$, satisfying $u = 0 = v$ on $\partial\Omega \cap B(w, r)$.

Theorem (A.-Lundström (2014))

(a) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b.

(b) Let $w \in \partial\Omega$, $0 < r < r_b$.

(c) Let p be a bounded Lipschitz continuous variable exponent.

Assume that u and v are positive $p(\cdot)$-harmonic functions in $\Omega \cap B(w, r)$, satisfying $u = 0 = v$ on $\partial\Omega \cap B(w, r)$.
Then there exist constants C and $\tilde{c}$ such that

$$\frac{1}{C} \leq \frac{u(x)}{v(x)} \leq C \qquad \text{for} \quad x \in \Omega \cap B(w, r/\tilde{c}).$$

(a) Constant $\tilde{c}$ depends on r_b and $p^-, ||\nabla p||_{L^\infty}$,

Boundary Harnack inequality: the $p(\cdot)$-Laplacian

Theorem (A.-Lundström (2014))

(a) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b.

(b) Let $w \in \partial\Omega$, $0 < r < r_b$.

(c) Let p be a bounded Lipschitz continuous variable exponent.

Assume that u and v are positive $p(\cdot)$-harmonic functions in $\Omega \cap B(w, r)$, satisfying $u = 0 = v$ on $\partial\Omega \cap B(w, r)$.
Then there exist constants C and $\tilde{c}$ such that

$$\frac{1}{C} \leq \frac{u(x)}{v(x)} \leq C \qquad \text{for} \quad x \in \Omega \cap B(w, r/\tilde{c}).$$

(a) Constant $\tilde{c}$ depends on r_b and $p^-, ||\nabla p||_{L^\infty}$,

(b) Constant C depends on $n, p^+, p^-, ||\nabla p||_{L^\infty}, r_b$, and constants from the Carleson estimate.

Theorem (A.-Lundström (2014))

(a) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b.

(b) Let $w \in \partial\Omega$, $0 < r < r_b$.

(c) Let p be a bounded Lipschitz continuous variable exponent.

Assume that u and v are positive $p(\cdot)$-harmonic functions in $\Omega \cap B(w, r)$, satisfying $u = 0 = v$ on $\partial\Omega \cap B(w, r)$.
Then there exist constants C and $\tilde{c}$ such that

$$\frac{1}{C} \leq \frac{u(x)}{v(x)} \leq C \qquad \text{for} \quad x \in \Omega \cap B(w, r/\tilde{c}).$$

(a) Constant $\tilde{c}$ depends on r_b and $p^-, ||\nabla p||_{L^\infty}$,

(b) Constant C depends on $n, p^+, p^-, ||\nabla p||_{L^\infty}, r_b$, and constants from the Carleson estimate.

Theorem (A.-Lundström (2014))

(a) Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b.

(b) Let $w \in \partial\Omega$, $0 < r < r_b$.

(c) Let p be a bounded Lipschitz continuous variable exponent.

Assume that u and v are positive $p(\cdot)$-harmonic functions in $\Omega \cap B(w, r)$, satisfying $u = 0 = v$ on $\partial\Omega \cap B(w, r)$.
Then there exist constants C and $\tilde{c}$ such that

$$\frac{1}{C} \leq \frac{u(x)}{v(x)} \leq C \qquad \text{for} \quad x \in \Omega \cap B(w, r/\tilde{c}).$$

(a) Constant $\tilde{c}$ depends on r_b and $p^-, ||\nabla p||_{L^\infty}$,

(b) Constant C depends on $n, p^+, p^-, ||\nabla p||_{L^\infty}, r_b$, and constants from the Carleson estimate.

Boundary Harnack inequality: main ingredients of the proof

(1) The Harnack inequality for $p(\cdot)$-harmonic functions.

(2) Variable exponent Carleson-type estimate for NTA-domains (in particular satisfying the ball condition)

$$\sup_{\Omega \cap B(w,r)} u \leq c\left(u(a_r(w)) + r\right),$$

where $a_r(w) \in \Omega$ is the point depends on geometric parameters of the domain and $w \in \partial\Omega$.

Boundary Harnack inequality: main ingredients of the proof

(1) The Harnack inequality for $p(\cdot)$-harmonic functions.

(2) Variable exponent Carleson-type estimate for NTA-domains (in particular satisfying the ball condition)

$$\sup_{\Omega \cap B(w,r)} u \leq c\left(u(a_r(w)) + r\right),$$

where $a_r(w) \in \Omega$ is the point depends on geometric parameters of the domain and $w \in \partial\Omega$.

(3) Barrier functions.

Boundary Harnack inequality: main ingredients of the proof

(1) The Harnack inequality for $p(\cdot)$-harmonic functions.

(2) Variable exponent Carleson-type estimate for NTA-domains (in particular satisfying the ball condition)

$$\sup_{\Omega\cap B(w,r)} u \leq c\left(u(a_r(w)) + r\right),$$

where $a_r(w) \in \Omega$ is the point depends on geometric parameters of the domain and $w \in \partial\Omega$.

(3) Barrier functions.

Thank you for your attention!