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# Convolution type operators with symmetry

# Contents

# 1 Introduction

***Convolution type operators with symmetry*** in spaces of Bessel potentials

$$T = r_+ A \ell^c : H^r(\mathbb{R}_+) \to H^s(\mathbb{R}_+) \tag{1}$$

where $r, s \in \mathbb{R}$ and [CST04]

$$\begin{aligned} \ell^c &: H^r(\mathbb{R}_+) \to H^r = H^r(\mathbb{R}) \text{ cont. extension op. (e/o)} \\ r_+ &: H^s(\mathbb{R}) \to H^s(\mathbb{R}_+) \qquad \text{restriction operator} \\ A &= \mathcal{F}^{-1}\phi \cdot \mathcal{F} : H^r \to H^s \text{ convolution op.} \end{aligned}$$

with measurable Fourier symbol $\phi$ of order $r - s$ :

$$(\lambda^{s-r}\phi)^{\pm 1} \in L^\infty(\mathbb{R}), \text{ where } \lambda(\xi) = (\xi^2 + 1)^{1/2}, \ \xi \in \mathbb{R}.$$

Furthermore, for $r = s$ let $\ell^c$ be left invertible by $r_+$ :

$$r_+ \ell^c = I|_{H^r(\mathbb{R}_+)}, \quad \ell^c r_+ = P \text{ a projector along } H^r_-.$$

***Prototypes***

$$W \quad f(x) = af(x) + \int_0^\infty k(x-y)f(y)dy = g(x) \ , \ \ x \in \mathbb{R}_+$$

classical Wiener-Hopf operator in $L^2(\mathbb{R}_+)$ , $(r = s = 0)$

$W = r_+ A\ell_0 \ : \ H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$ WHO , $s \in ]-1/2, +1/2[$

$T = r_+ A\ell^e = r_+ A(I+J)\ell_0$ WH+HO $s \in ]-1/2, +3/2[$

$T = r_+ A\ell^o = r_+ A(I-J)\ell_0$ WH-HO $s \in ]-3/2, +1/2[$

$W = r_+ A|_{H^r_+} \ : \ H^r_+ \to H^s(\mathbb{R}_+)$ Eskin notation , $r, s \in \mathbb{R}$

$W = P_2 A_{|P_1 X}$ General WHO

Typically in this context:

$$P_j = \ell^c r_+$$

with different extension operators for $j = 1$ and $j = 2$.

***A glance at general WHOs: the standard setting***

$$W = P_2 A_{|P_1 X} \;:\; P_1 X \to P_2 Y$$

where

$$\begin{aligned} X, Y &: \text{ Banach spaces }, \text{ here Hilbert spaces} \\ P_1 &= P_1^2 \in \mathcal{L}(X) \;,\quad P_2 = P_2^2 \in \mathcal{L}(Y) \quad \text{projectors} \\ A &\in \mathcal{L}(X, Y) \quad \text{boundedly invertible} \end{aligned} \tag{2}$$

Only $\operatorname{im} P_1 = P_1 X$ and $\ker P_2 = (I - P_2) Y$ are relevant:

$$\begin{aligned} W &= P_2 A_{|\tilde{P}_1 X} \text{ if } \tilde{P}_1 = \tilde{P}_1^2 \in \mathcal{L}(X) \,,\; P_1 \tilde{P}_1 = \tilde{P}_1 \,,\; \tilde{P}_1 P_1 = P_1 \\ &\sim \tilde{P}_2 A_{|\tilde{P}_1 X} \text{ if } \tilde{P}_2 = \tilde{P}_2^2 \in \mathcal{L}(X) \,,\; P_2 \tilde{P}_2 = P_2 \,,\; \tilde{P}_2 P_2 = \tilde{P}_2 \end{aligned}$$

### *Variants of projectors: basic cases*

| projector onto → along ↓ | $L^2_+$ | $L^2_-$ | $L^2_e$ | $L^2_o$ |
|---|---|---|---|---|
| $L^2_+$ | - | $P_- = \ell_0 r_-$ | $(I+J)P_-$ | $(I-J)P_-$ |
| $L^2_-$ | $P_+ = \ell_0 r_+$ | - | $(I+J)P_+$ | $(I-J)P_+$ |
| $L^2_e$ | $P_+(I-J)$ | $P_-(I-J)$ | - | $\frac{1}{2}(I-J)$ |
| $L^2_o$ | $P_+(I+J)$ | $P_-(I+J)$ | $\frac{1}{2}(I+J)$ | - |

**Remarks:** 1. $JP_+ = P_-J \ , \ JP_- = P_+J \ , \ JA_\phi = A_{\tilde{\phi}}J \ .$
2. Projectors restricted/extended to $H^s$, $|s| < 1/2$, are continuous.

***Type of CTOS*** $T = r_+ A\ell^c \sim P_2 A_{|P_1 X}$

| projector $P_1$ onto $\rightarrow$ $P_2$ along $\downarrow$ | $L^2_+$ | $L^2_-$ | $L^2_e$ | $L^2_o$ |
|---|---|---|---|---|
| $L^2_-$ | WHO | HO | WH+HO | WH-HO |
| $L^2_+$ | HO | WHO | WH-HO | WH+HO |
| $L^2_o$ | WH+HO | WH-HO | CO even symbol | CO odd symbol |
| $L^2_e$ | WH-HO | WH+HO | CO odd symbol | CO even symbol |

HO = Hankel operator , CO = Convolution operator (just restricted) and possible continuous extension/restriction to $H^s$, $s \in \mathbb{R}$.

***First Questions***

How to invert? Efficiently? For all $s \in \mathbb{R}$ ?

Other kinds of symmetry? Relevant?

Applications?

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# 2 Even and odd extension

For any $r \in \mathbb{R}$, let

$$H^{r,e} = \{\varphi \in H^r : \varphi = J\varphi\} \ , \qquad H^{r,o} = \{\varphi \in H^r : \varphi = -J\varphi\} \qquad (3)$$

where $J\varphi(x) = \varphi(-x)$ for $\varphi \in H^r$, $r \geq 0$, and $J\varphi(\psi) = \varphi(J\psi)$ for test functions $\psi$ in the case of $r < 0$, respectively.

**Lemma 2.1** [MSPT93,CST04] *The operator of even extension $\ell^e$ is a well-defined and bounded operator from $H^r(\mathbb{R}_+)$ into $H^r$ with the image $H^{r,e}$ if and only if $r \in ]-1/2, 3/2[$. In this case, $r_+\ell^e = I$ and $\ell^e r_+$ is a bounded projection in $H^r$ with the image $H^{r,e}$. Otherwise, $\ell^e$ does not map into $H^{r,e}$ $(r \geq 3/2)$ or not onto $H^{r,e}$ $(r \leq -1/2)$, respectively.*

**Lemma 2.2** [MSPT93,CST04] *The operator of odd extension $\ell^o$ is a well-defined and bounded operator from $H^r(\mathbb{R}_+)$ into $H^r$ with image $H^{r,o}$ if and only if $r \in ]-3/2, 1/2[$. In that case, $r_+\ell^o = I$ and $\ell^o r_+$ is a bounded projection in $H^r$ with the image $H^{r,o}$. Otherwise, $\ell^o$ does not map into $H^{r,o}$ $(r \geq 1/2)$ or not onto $H^{r,o}$ $(r \leq -3/2)$, respectively.*

**Proposition 2.3** *Let* $r \in ]-1/2,\, 3/2[$ *and* $\ell^c = \ell^e$ *(or* $r \in ]-3/2,\, 1/2[$ *and* $\ell^c = \ell^o$*). Then the operator* (1) *can be* lifted into $L^2(\mathbb{R}_+)$*, i.e., there are homeomorphisms* $E$*,* $F$ *such that*

$$\begin{aligned} T &= E\, T_0\, F \\ T_0 &= r_+ A_0 \ell^c \;:\; L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \\ A_0 &= \mathcal{F}^{-1} \phi_0 \cdot \mathcal{F} \end{aligned} \tag{4}$$

*where* $\phi_0 \in \mathcal{G}L^\infty(\mathbb{R})$*, i.e.,* $\phi_0$ *is invertible in* $L^\infty(\mathbb{R})$*, provided* $A$ *is bijective. More precisely, let* $\lambda_-(\xi) = \xi - i$*,* $\xi \in \mathbb{R}$*,* $A_{\lambda_-^s} = \mathcal{F}^{-1} \lambda_-^s \cdot \mathcal{F}$ *and* $A_{\lambda^{-r}} = \mathcal{F}^{-1} \lambda^{-r} \cdot \mathcal{F}$ *where* $\lambda(\xi) = (\xi^2 + 1)^{1/2}$*; then we can take*

$$\begin{aligned} A_0 &= A_{\lambda_-^s} A A_{\lambda^{-r}} \\ \phi_0 &= \lambda_-^s\, \phi\, \lambda^{-r} \\ E &= r_+ A_{\lambda_-^{-s}} \ell \\ F &= r_+ A_{\lambda^r} \ell^c \end{aligned} \tag{5}$$

*where* $\ell : L^2(\mathbb{R}_+) \to L^2 = L^2(\mathbb{R})$ *denotes an arbitrary extension, i.e.,* $E$ *is independent of that choice.*

# 3 Asymmetric generalized factorization

Let $L^2_-$ be the image in $L^2$ of the projection $P_- = (I - S_{\mathbb{R}})/2$ due to the Hilbert transformation $S_{\mathbb{R}}$. For a subspace $X$ of $L^2$, $X(\mathbb{R}, \rho)$ denotes the weighted space whose elements $\varphi$ fulfill $\rho\varphi \in X$.

**Definition 3.1** *A function $\phi \in \mathcal{G}L^\infty(\mathbb{R})$ admits an* asymmetric generalized factorization with respect to $L^2$ and $\ell^e$ *(AGF), written as*

$$\phi = \phi_- \, \zeta^\kappa \, \phi_e \tag{6}$$

*if $\kappa \in \mathbb{Z}$, $\zeta(\xi) = (\xi - i)/(\xi + i)$ for $\xi \in \mathbb{R}$, $\phi_- \in L^2_-(\mathbb{R}, \lambda_-^{-2})$, $\phi_-^{-1} \in L^2_-(\mathbb{R}, \lambda_-^{-1})$, $\phi_e \in L^{2,e}(\mathbb{R}, \lambda^{-1})$, $\phi_e^{-1} \in L^{2,e}(\mathbb{R}, \lambda^{-2})$ and if*

$$V = A_e^{-1} \ell^c r_+ C^{-1} \ell^c r_+ A_-^{-1} \tag{7}$$

*is bounded in $L^2$ for $\ell^c = \ell^e$ (as a composition of unbounded operators, extended from a dense subspace). As usual the factor spaces are the closures of the spaces of bounded rational functions without poles in the closed lower half-plane $\overline{\mathbb{C}_-} = \{\xi \in \mathbb{C} : \Im m(\xi) \leq 0\}$ or of those which are even, respectively, due to the weighted $L^2$ norm. For brevity, we put $A_e = \mathcal{F}^{-1}\phi_e \cdot \mathcal{F}$, $A_- = \mathcal{F}^{-1}\phi_- \cdot \mathcal{F}$ and $C = \mathcal{F}^{-1}\zeta^\kappa \cdot \mathcal{F}$.*

*More precisely, the domains of the factors are such that*

$$A = A_- C A_e : \mathcal{D}_1 \to \mathcal{D} \to \mathcal{D} \to \mathcal{D}_2 \tag{8}$$

*splits in the sense of bijective mappings where*

$$\begin{aligned} \mathcal{D}_1 &= \mathcal{D}(A_e) \\ \mathcal{D}_2 &= \mathcal{D}(A_-^{-1}) \\ \mathcal{D} \ &= \ \operatorname{im} A_{e|\mathcal{D}_1} = \ \operatorname{im} A_-^{-1}{}_{|\mathcal{D}_2}\,, \end{aligned} \tag{9}$$

*$\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}$ are dense in $L^2$ and the restrictions of $C^{\pm 1}$, $\ell^c r_+ : \mathcal{D} \to \mathcal{D}$ are well-defined. Note that the weights $\lambda_-^{-1}$, $\lambda^{-1}$ have the familiar decay at infinity that is used for generalized factorization in $L^2(\mathbb{R})$ whilst the weights $\lambda_-^{-2}$, $\lambda^{-2}$ admit an increase of the factors $\phi_-$ and $\phi_e^{-1}$, respectively, that is one order higher than usual.*

Further we speak of an AGF with respect to $L^2$ and $\ell^o$ if (6) holds with $\phi_- \in L^2(\mathbb{R}, \lambda_-^{-1})$, $\phi_-^{-1} \in L^2_-(\mathbb{R}, \lambda_-^{-2})$, $\phi_e \in L^{2,e}(\mathbb{R}, \lambda^{-2})$, $\phi_e^{-1} \in L^{2,e}(\mathbb{R}, \lambda^{-1})$ and if (7) is bounded in $L^2$ for $\ell^c = \ell^o$.

**Theorem 3.2** *Let*

$$T = r_+ A\ell^c : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \tag{10}$$

*where* $A = \mathcal{F}^{-1}\phi \cdot \mathcal{F} \in \mathcal{GL}(L^2)$, *i.e.,* $\phi \in \mathcal{G}L^\infty$, *and let* $\phi$ *have an AGF* (6). *Then* $T$ *is generalized invertible, i.e., there exists an operator* $T^- \in \mathcal{L}\left(L^2(\mathbb{R}_+)\right)$ *such that* $TT^-T = T$. *Such a generalized inverse* $T^-$ *of* $T$ *is given by*

$$T^- = r_+ V\ell = r_+ A_e^{-1} \ell^c r_+ C^{-1} \ell^c r_+ A_-^{-1} \ell\,. \tag{11}$$

*Moreover this is a reflexive generalized inverse of* $T$, *i.e.,* $T^-TT^- = T^-$ *holds as well.*

**The direct method.**

Now we come to the constructive factorization of scalar symbols of normal type and confine ourselves to a class which is most relevant for the applications mentioned before, namely elements of the algebra of Hölder continuous functions on $\ddot{\mathbb{R}} = [-\infty, +\infty]$:

$$C^\nu(\ddot{\mathbb{R}}) = \Big\{ \phi \in C^\nu(\mathbb{R}) : \phi(\pm\infty) = \lim_{\xi\to\pm\infty} \phi(\xi) \text{ exist and}$$
$$\phi(\xi) - \phi(\pm\infty) = \mathcal{O}\Big(|\xi|^{-\nu}\Big) \text{ as } \xi \to \pm\infty \Big\}, \qquad \nu \in ]0,1[\,.$$

In the case of $\phi(+\infty) = \phi(-\infty)$ we write $\phi \in C^\nu(\dot{\mathbb{R}})$. The subclasses of regular elements are denoted by $\mathcal{G}C^\nu(\ddot{\mathbb{R}})$ and $\mathcal{G}C^\nu(\dot{\mathbb{R}})$, respectively.

**Proposition 3.3** *Let* $\phi \in \mathcal{G}C^{\nu}(\ddot{\mathbb{R}})$ *where* $\nu \in ]0,1[$ *and define*

$$\omega = \frac{1}{2\pi i}\int_{\mathbb{R}} d\,\log\phi\,. \tag{12}$$

*Further let*

$$\Re e(\omega) \pm \frac{1}{4} \notin \mathbb{Z} \tag{13}$$

*where the sign corresponds with the* $\ell^e/\ell^o$ *case. Then* $\phi$ *admits an AGF given by the following formulas:*

$$\begin{aligned}
\kappa &= \max\left\{z \in \mathbb{Z} : z \leq \Re e(\omega) \pm \frac{1}{4}\right\} \\
\psi &= \zeta^{-\omega}\,\phi^{-1}(+\infty)\ \phi \\
\widetilde{\psi} &= J\psi \\
\phi_- &= \lambda_-^{2(\omega-\kappa)}\exp\left\{P_-\log\left(\psi\widetilde{\psi}^{-1}\right)\right\} \\
\phi_e &= \zeta^{-\kappa}\phi_-^{-1}\phi\,.
\end{aligned} \tag{14}$$

***Sketch of the proof***

We write $\phi$ in the form

$$\phi(\xi) = \phi(+\infty)\left(\frac{\xi - i}{\xi + i}\right)^{\omega} \psi(\xi)\,, \quad \xi \in \mathbb{R} \tag{15}$$

where $\psi \in \mathcal{G}C^{\nu}(\dot{\mathbb{R}})$ with vanishing winding number. Precisely, let $\omega = \sigma + i\tau \in \mathbb{C}$ with real and imaginary parts given by

$$\sigma = \frac{1}{2\pi}\int_{\mathbb{R}} d\,\arg\phi\,, \quad \tau = \frac{1}{2\pi}\log\left|\frac{\phi(-\infty)}{\phi(+\infty)}\right|\,. \tag{16}$$

Now we follow an idea of Basor and Ehrhardt from the theory of Toeplitz plus Hankel operators and consider the function

$$G = \psi\,\widetilde{\psi}^{-1}. \tag{17}$$

It has the same properties as $\psi$ before plus the (anti–) symmetry property $\widetilde{G}^{-1} = G$. Thus it admits a canonical *anti–symmetric factorization*

$$G = G_- G_+ = G_- \widetilde{G}_-^{-1} \tag{18}$$

where $G_\pm \in \mathcal{G}C^{\nu}_{\pm}(\dot{\mathbb{R}})$.

By the help of (3) and (18) we obtain an AGF of $\psi$, putting

$$\psi = \psi_- \psi_e = G_- \psi_e \,. \tag{19}$$

Here $\psi_e$ is even since this fact is equivalent to

$$\begin{aligned} \psi_e &= \widetilde{\psi}_e \\ \psi\, G_-^{-1} &= \widetilde{\psi}\, \widetilde{G}_-^{-1} \\ \psi\, \widetilde{\psi}^{-1} &= G_-\, \widetilde{G}_-^{-1} \end{aligned} \tag{20}$$

which was our factorization (18) of (17). The factors of $\psi$ in (19) belong to $\mathcal{G}C^\nu(\dot{\mathbb{R}})$. Therefore $\phi$ has an AGF if and only if $\zeta^\omega$ admits an AGF.

Now let (13) be satisfied, thus we can write

$$\omega = \sigma + i\tau = \kappa + \eta + i\tau \tag{21}$$

where $\kappa \in \mathbb{Z}$ and $\eta \in ]-1/4, 3/4[$ in the case $\ell^c = \ell^e$, that will be treated first. Considering

$$\begin{aligned} \zeta^\omega &= \lambda_-^{2(\omega-\kappa)} \zeta^\kappa (\lambda_- \lambda_+)^{\kappa-\omega} \\ &= \lambda_-^{2(\eta+i\tau)} \zeta^\kappa \lambda^{-2(\eta+i\tau)} \end{aligned} \tag{22}$$

we have an AGF with respect to $\ell^e$: the factors belong to the spaces defined after (6) and the operator $V$ in (7) corresponding with (22) is bounded. This is a consequence of the fact that, due to (22),

$$C_\omega = \mathcal{F}^{-1} \zeta^\omega \cdot \mathcal{F} = C_- C C_e : L^2 \to H^{2\eta} \to H^{2\eta} \to L^2 \tag{23}$$

is a composition of boundedly invertible operators where $2\eta \in ]-1/2, 3/2[$, such that $\ell^e r_+$ is bounded in $H^{2\eta}$, see Lemma 2.1. I.e., the factors $C_-$, $C$, $C_e$ are bijections with respect to the spaces mentioned in (23). Evidently, the combination with (15) and (19) yields an AGF of $\phi$.

The case $\ell^c = \ell^o$ runs analogously with $\eta \in ]-3/4, 1/4[$ and $2\eta \in ]-3/2, 1/2[$. **End of Proof**

**Corollary 3.4** *Under the same assumptions* (12)–(13) *as before, the AGF of* $\phi$ *yields* bounded operator factorizations *of the multiplication operator* $\phi\cdot$, *of the convolution operator* $A = \mathcal{F}^{-1}\phi\cdot\mathcal{F}$ *and of the Wiener-Hopf plus Hankel operator* $T = r_+ A\ell^e$ through an intermediate space $Z$ *which is a weighted* $L^2$ *space or corresponding Sobolev space, respectively. More precisely, we have a commutative diagram of bijective multiplication operators*

$$\begin{array}{ccc} L^2(\mathbb{R}) & \overset{\phi\,\cdot}{\longrightarrow} & L^2(\mathbb{R}) \\ \phi_e\cdot \downarrow & & \uparrow \phi_-\cdot \\ L^2(\mathbb{R}, \lambda^{-2\eta}) & \underset{\zeta^\kappa\cdot}{\longrightarrow} & L^2(\mathbb{R}, \lambda^{-2\eta}) \end{array} \tag{24}$$

*and, correspondingly written as bounded operator factorization through an intermediate space [CS95,S14]*

$$A = A_- C A_e : L^2 \to H^{2\eta} \to H^{2\eta} \to L^2 \tag{25}$$

$$\begin{aligned} T &= r_+ A_- \ell\,(r_+ C \ell^e)\, r_+ A_e \ell^e \qquad (26) \\ &: \quad L^2(\mathbb{R}_+) \to H^{2\eta}(\mathbb{R}_+) \to H^{2\eta}(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \end{aligned}$$

*(where the extension $\ell : H^{2\eta}(\mathbb{R}_+) \to H^{2\eta}$ is arbitrary, e.g. $\ell = \ell^e$ or $\ell = \ell^o$).*

*Furthermore the intermediate space $Z = H^{2\eta}$ in (25) has the following properties, for all possible choices of $\omega$,*

$$\begin{aligned} & C = \zeta^\kappa \cdot \in \mathcal{L}(Z) \qquad \textit{for all} \quad \kappa \in \mathbb{Z} \\ & \ell^e r_+ \in \mathcal{L}(Z)\,. \end{aligned} \qquad (27)$$

*Finally, denoting the $L^2$ functions with rational Fourier images by $\widehat{\mathcal{R}}_0 = \widehat{\mathcal{R}} \cap L^2(\mathbb{R})$, we have*

$$\ell^e r_+ \widehat{\mathcal{R}}_0 \underset{\text{dense}}{\subset} \ell^e r_+ Z\,. \qquad (28)$$

# 4 Fredholm and invertibility properties

Many results are analogous to WHOs! Here are some of them:

**Theorem 4.1** *Let* $\phi \in \mathcal{G}C^{\nu}(\ddot{\mathbb{R}})$, $\nu \in ]0,1[$ *and* $\omega$ *be given by* (12) *and* $T = r_+ A\ell^c = r_+ \mathcal{F}^{-1}\phi \cdot \mathcal{F}\ell^c : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$. *Then the following conditions are equivalent:*

*(i)* $\Re e(\omega) \pm \frac{1}{4} \notin \mathbb{Z}$,

*(ii)* $\phi$ *admits an AGF* (6) *with respect to* $L^2$ *and* $\ell^e$ *or* $\ell^o$, *respectively,*

*(iii)* $T$ *is Fredholm,*

*and, moreover,*

*(ii′)* $A$ *admits an AFIS* (25) *through the space* $Z = H^{2\eta}$ *where* $2\eta \in ]-1/2, 3/2[$ *or* $2\eta \in ]-3/2, 1/2[$, *respectively.*

*Further the factorization of* $\phi$ *is unique up to constant factors in* $\phi_-$ *and* $\phi_e$ *(inverse to each other).*

**Corollary 4.2** *Consider again the operator*

$$T = r_+ A \ell^c : H^r(\mathbb{R}_+) \to H^s(\mathbb{R}_+) \tag{29}$$

*where* $A = \mathcal{F}^{-1}\phi \cdot \mathcal{F} : H^r \to H^s$ *is a bijection,* $s \in \mathbb{R}$ *and either*

$$\ell^c = \ell^e \,, \qquad r \in \left]-\frac{1}{2}, \frac{3}{2}\right[ \tag{30}$$

*or*

$$\ell^c = \ell^o \,, \qquad r \in \left]-\frac{3}{2}, \frac{1}{2}\right[ \,. \tag{31}$$

*Then* $T$ *is Fredholm if and only if*

$$\sigma = \frac{1}{2\pi}\int_{\mathbb{R}} d \arg \phi_0 \neq \mp \frac{1}{4} \qquad \mod \mathbb{Z} \tag{32}$$

*where* $\phi_0 = \lambda_-^s \phi \lambda^{-r}$.

In this case

$$\operatorname{ind} T = -\kappa = \begin{cases} \alpha(T) & \text{if} \quad \kappa \leq 0 \\ -\beta(T) & \text{if} \quad \kappa \geq 0 \end{cases} \tag{33}$$

and a reflexive generalized inverse of $T$ is given by

$$\begin{aligned} T^- &= r_+ A_{\lambda^{-r}} A_e^{-1} \ell^e r_+ C^{-1} \ell^e r_+ A_-^{-1} A_{\lambda_-^s} \ell \\ &: H^s(\mathbb{R}_+) \to H^{2\eta}(\mathbb{R}_+) \to H^{2\eta}(\mathbb{R}_+) \to H^r(\mathbb{R}_+) \end{aligned} \quad (34)$$

where $A_0 = \mathcal{F}^{-1}\phi_0 \cdot \mathcal{F} = A_- C A_e$ is a factorization as in (25), $\eta$ being defined by (21).

$T$ is invertible if and only if, moreover, $\kappa = 0$ and $C = I$ and we have $T^{-1} = T^-$ in this case.

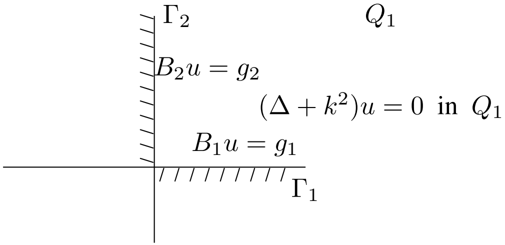
**Remark:** All these results can be generalized for

$$T = r_+ A|_{H^{r,c}} : H^{r,c} \to H^s(\mathbb{R}_+) \ , \ \ r, s \in \mathbb{R}$$

if (32) is satisfied and by ”image normalization” to arbitrary $\omega \in \mathbb{C}$.

# 5 Some applications

All the work is highly motivated by so-called canonical diffraction problems concerned with linear wave propagation and wedge-shaped obstacles.




**Example:** Weak solution of the (mixed) Dirichlet-Neumann problem

$$\begin{aligned} L \quad : \quad \mathcal{H}^1(Q_1) &\to H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+) \\ u &\mapsto (u|_{\Gamma_1}, \frac{\partial u}{\partial n}|_{\Gamma_2}) \quad = \quad (g_1, g_2) \end{aligned}$$

The solution is amazingly simple in the DN case:

$$u(x_1,x_2) = \mathcal{F}^{-1}_{\xi\mapsto x_1} e^{-t(\xi)x_2}\widehat{\ell^e g_1}(\xi) - \mathcal{F}^{-1}_{\xi\mapsto x_2} e^{-t(\xi)x_1} t^{-1}(\xi)\widehat{\ell^o g_2}(\xi)$$

where $\ell^e$ and $\ell^o$ denote even and odd extension.

**Theorem** [MPST93, CST04]
The following mapping

$$\mathcal{K}_{DN,Q_1} : X = H^{1/2}(\Gamma_1)\times H^{-1/2}(\Gamma_2) \to \mathcal{H}^1(Q_1)$$

$$u = \mathcal{K}_{DN,Q_1}(f,g)^T = \mathcal{K}_{D,Q_{12}}\ell^e f + \mathcal{K}_{N,Q_{14}}\ell^o g$$

$$\mathcal{K}_{D,Q_{12}}\ell^e f(x) = \mathcal{F}^{-1}_{\xi\mapsto x_1} \exp\left[-t(\xi)x_2\right]\widehat{\ell^e f}(\xi)\ , \qquad x\in Q_{12}$$

$$\mathcal{K}_{N,Q_{14}}\ell^o g(x) = -\mathcal{F}^{-1}_{\xi\mapsto x_2} \exp\left[-t(\xi)x_1\right] t^{-1}(\xi)\,\widehat{\ell^o g}(\xi)\ , \quad x\in Q_{14}$$

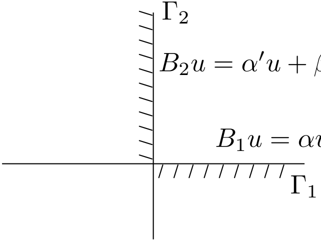
is a toplinear isomorphism that satisfies

$$(T_{0,\Gamma_1}, T_{1,\Gamma_2})^T\mathcal{K}_{DN,Q_1} = I_X$$

$$\mathcal{K}_{DN,Q_1}(T_{0,\Gamma_1}, T_{1,\Gamma_2})^T = I_{\mathcal{H}^1(Q_1)}\ .$$

- Using this representation as a potential operator, it was possible to solve explicitly a great number of BVPs, see CST 2004, because the corresponding boundary ΨDO a matricial $2 \times 2$ structured operator that has a triangular form in many cases.
- A more sophisticated ansatz $u = \mathcal{K}_{B_2 * B_2, Q_1}(f_1, f_2)$ (with so-called *half-line potentials*), see CST 06, led to the following structured operator matrices $T \sim L$ for the operator associated to the full BVP

$$(\Delta + k^2)u = 0 \text{ in } Q_1$$




$$B_2 u = \alpha' u + \beta' \frac{\partial u}{\partial x_1} + \gamma' \frac{\partial u}{\partial x_2} = g_2 \text{ on } \Gamma_2$$

$$B_1 u = \alpha u + \beta \frac{\partial u}{\partial x_2} + \gamma \frac{\partial u}{\partial x_1} = g_1 \text{ on } \Gamma_1$$

**Proposition** Let $L = (B_1, B_2)^T$ and $\mathcal{K}$ be given as before. Then the composed operator $T = L\mathcal{K}$ has the form:

$$T = \begin{pmatrix} r_+ A_{\phi_{11}} \ell_1 & C_0 A_{\phi_{12}} \ell_2 \\ C_0 A_{\phi_{21}} \ell_1 & r_+ A_{\phi_{22}} \ell_2 \end{pmatrix} : X \to Y$$

where $Y = H^{-1/2}(\mathbb{R}_+)^2$ identifying $\Gamma_j$ with $\mathbb{R}_+$ and

$$\phi_{11} = \sigma_1 \psi_1^{-1} = (\alpha - \beta t + \gamma\vartheta)\psi_1^{-1}\ , \quad \phi_{12} = \sigma_{1*}\psi_2^{-1} = (\alpha + \beta\vartheta - \gamma t)\psi_2^{-1}$$
$$\phi_{21} = \sigma_{2*}\psi_1^{-1} = (\alpha' + \beta'\vartheta - \gamma' t)\psi_1^{-1}\ , \quad \phi_{22} = \sigma_2 \psi_2^{-1} = (\alpha' - \beta' t + \gamma'\vartheta)\psi_2^{-1}$$

The main diagonal contains CTOS if $\ell_j = \ell^{e/o}$. The others are Fourier integral operators (combined with extensions) defined for any $\phi \in L^\infty$ by

$$K^{(s)} = C_0 A_\phi \ell_j : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$$
$$K^{(s)} f(x_1) = (2\pi)^{-1} \int_{\mathbb{R}} \exp[-t(\xi)x_1] \phi(\xi) \widehat{\ell_j f}(\xi)\, d\xi\ , \quad x_1 \in \mathbb{R}_+\ .$$

They are well-defined and bounded if $s \in ]-3/2, 1/2[$ or $s \in ]-1/2, 3/2[$, resp.

**Strategy for the generalized inversion of $T$**

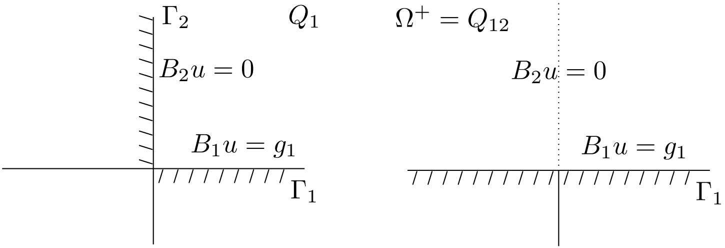
- $T$ is triangular, if one of the off-diagonal operators $K^{(s)} = 0$ (i.e., if $\phi$ is an even function in case $\ell^o$ and if $\phi$ is an odd function in case $\ell^e$).
- The DN ansatz yields triangular $T$ in plenty cases.
- The companion operator ansatz* covers all others! Including:
- Oblique derivative problems: Fredholm operators $T$ with index $\pm 1$ appear!
- Two-impedance problems: The ansatz itself is only Fredholm, not invertible, but the problem is well-posed.

* Instead of the $DN$ problem solve the $B_1 B_1*$ problem or $B_2 * B_2$ problem first and then take the resolvent as a new representation formula.

Here we propose another approach via symmetry arguments.

# 6 Other forms of symmetry

Consider the following two boundary value problems for $u \in \mathcal{H}^1(Q_1)$ and for $u \in \mathcal{H}^1(Q_{12})$, respectively (after reduction to semi-homogeneous problems [S13]):




Questions: Under what conditions on $B_j$ :

Are they equivalent? Are they well-posed?

Then we can say that a solution of the first problem has a unique extension $u$ in $\mathcal{H}^1(Q_{12})$, the latter being "$B_2$-symmetric" with respect to $\Gamma_2$.

**Strategy and future music**

- The answer is YES if $B_2$ is Dirichlet $\to \ell^e$, or Neumann $\to \ell^o$.
- The method yields projectors of the form $A_\psi \ell^{e/o} A_\psi^{-1}$ reflecting "$B_2$-symmetry".
- Hence, in general only scalar problems and CTOS appear, moreover data correlations containing FIOs. Those yield (in several cases) necessary compatibility conditions which imply a minimal image normalization of $T$.

- It has to do with extension operators for solutions of the Helmholtz equation in a cone into a larger cone.
- This is related to the inversion of pure Hankel operators with special $\phi \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})$ symbols like $\zeta^\omega$.
- It implies results on the higher regularity (smoothness) of solutions in cones and the detection of exact necessary compatibility conditions on the data.

# 7 Further questions and open problems

- Mapping properties of FIOs
- High regularity, shift theorem
- Discovery of compatibility conditions (a priori)
- Normalization in spaces of higher order
- Connection with extension operators
- Inversion of pure Hankel operators

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