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Special Session on Homological and Combinatorial Commutative Algebra

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On the ideal of orthogonal representations of a graph in $\mathbb{R}^2$

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joint work with J. Herzog, S. Saeedi Madani, V. Welker

Orthogonal representations of a graph

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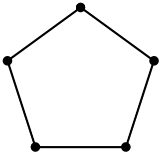

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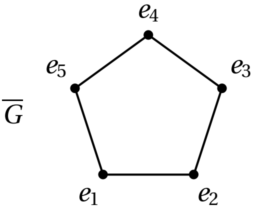

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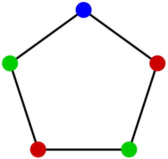

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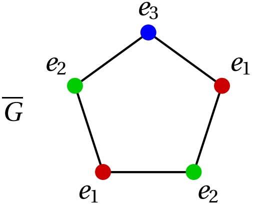

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Motivation: the Theta function and the Shannon capacity

The *theta function* (Lovász, 1979) of a graph G with vertices $[n]$ is

$$\theta(G) = \min_{(u_i),c} \max_{i \in [n]} \frac{1}{(c^T u_i)^2},$$

where the minimum is taken over all orthonormal representations $(u_i : i \in V)$ of G in $\mathbb{R}^d$, all unit vectors $c \in \mathbb{R}^d$ and integers $d \geq 1$.

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- $\theta(G)$ is polynomial time computable, while $\omega(\overline{G})$ and $\chi(\overline{G})$ are NP-complete, and the computational complexity of $\Theta(G)$ is unknown.

From Combinatorics to Commutative Algebra

From an algebraic point of view, the set of all orthogonal representations of a graph G is the vanishing set in $\mathbb{R}^{n\times d}$ of the ideal

$$L_{\overline{G}} = (x_{i1}x_{j1} + \cdots + x_{id}x_{jd} : \{i,j\} \in E(\overline{G}))$$

in the polynomial ring $\mathbb{R}[x_{ik} : i = 1, \ldots, n, k = 1, \ldots, d]$. We call $L_{\overline{G}}$ *Lovász-Saks-Schrijver ideal* of G.

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$$L_{\overline{G}} = (x_i x_j + y_i y_j : \{i, j\} \in E(\overline{G}))$$

as an ideal in the polynomial ring $T = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

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Let $d = 2$, $\sqrt{-1} \in K$ and G be a bipartite graph. Then L_G may be identified with the *binomial edge ideal* J_G of G.

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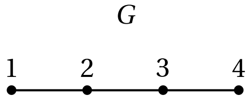

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Achtung! This identification does not hold for $K = \mathbb{R}$.

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The generators of Π_G are those 2-permanents of the matrix $\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$ whose column indices correspond to edges of G. Therefore, we call Π_G the *permanental edge ideal* of G.

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1.1 A Gröbner basis of Π_G

Let G be a graph with vertices $[n]$ and i, j be two distinct vertices of G. A *path* of length r in G from i to j is a sequence $\pi_{ij} : i = i_0, i_1, \ldots, i_r = j$ of pairwise distinct vertices such that $\{i_k, i_{k+1}\} \in E(G)$ for all k. We say that π_{ij} is *admissible* if $i < j$ and for each $k = 1, \ldots, r-1$, one has either $i_k < i$ or $i_k > j$.

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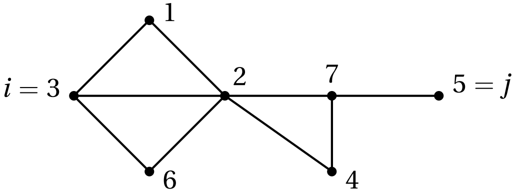

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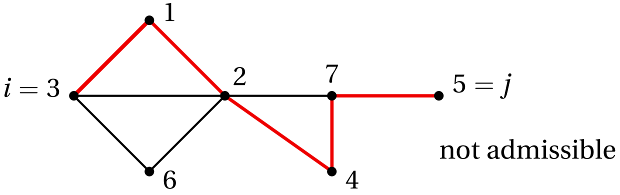

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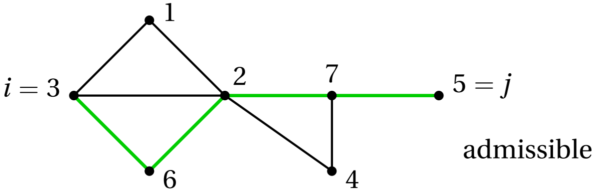

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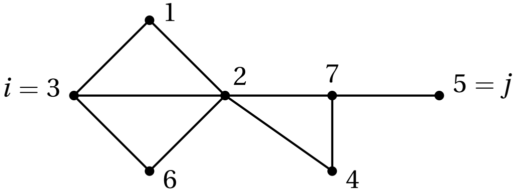

If π_{ij} is admissible, we attach to it the monomial

$$u_{\pi_{ij}} = \prod_{i_k > j} x_{i_k} \prod_{i_k < i} y_{i_k}.$$

Theorem (-) *Let G be a graph on $[n]$ and assume that $\operatorname{char}(K) \neq 2$. Then, with respect to the lexicographic order on $T = K[x_i, y_i]$ induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$, the following elements form a Gröbner basis of the ideal Π_G:*

1. *$u_{\pi_{ij}} b_{ij}$, where π_{ij} is an odd admissible path and $b_{ij} = x_i y_j + x_j y_i$,*
2. *$u_{\pi_{ij}} g_{ij}$, where π_{ij} is an even admissible path and $g_{ij} = x_i y_j - x_j y_i$,*
3. *$\operatorname{lcm}(u_{\pi_{ij}}, u_{\sigma_{ij}}) y_i x_j$, where π_{ij} is an odd and σ_{ij} is an even admissible path,*
4. $$\begin{cases} y_b \prod_{h \in W} x_h & \text{if } b < h \text{ for every } h \in W \\ x_b \prod_{h \in W} y_h & \text{if } b > h \text{ for every } h \in W \end{cases},$$
   where $W = V(\pi_{ij}) \cup V(\sigma_{ij}) \cup V(\tau_{ab}) \setminus \{b\}$, π_{ij} is an odd and σ_{ij} is an even admissible path from i to j, τ_{ab} is a path with endpoints a and b, such that a is the only vertex of τ_{ab} that belongs to $V(\pi_{ij}) \cup V(\sigma_{ij})$.

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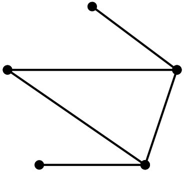

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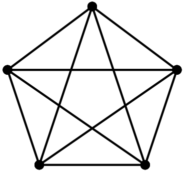

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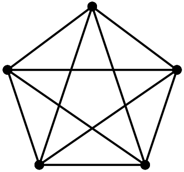

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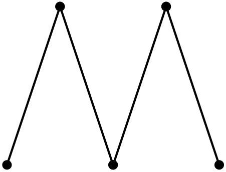

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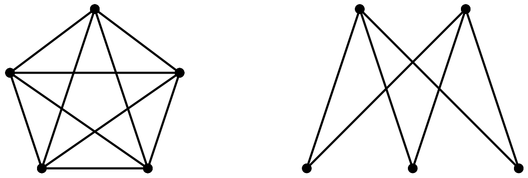

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Theorem (-) *Let G be a graph on $[n]$ and $\sqrt{-1} \notin K$. Then*

$$L_G = \bigcap_{S \subset [n]} Q_S(G)$$

is a redundant primary decomposition of L_G.

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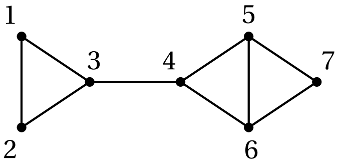

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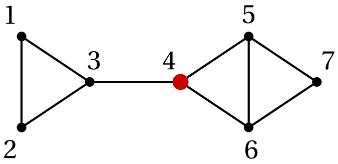

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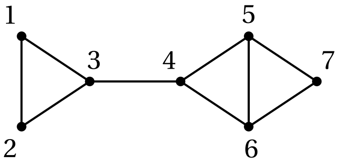

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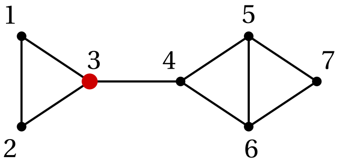

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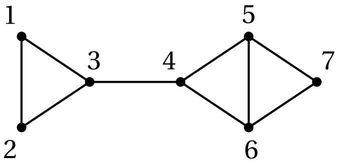

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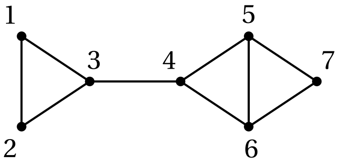

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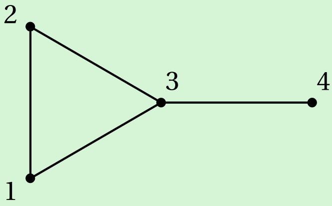
Theorem (-) *Let G be a graph on $[n]$, $\sqrt{-1} \in K$ and $S \subset [n]$. Then $Q_S(G)$ is a minimal prime ideal of L_G if and only if $S \in \mathcal{M}(G)$.*

Example Let G be the graph



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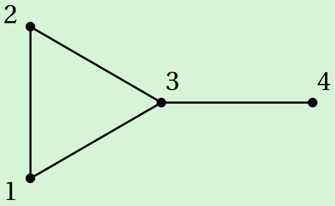
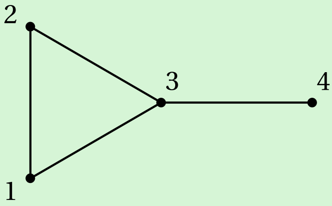


Then a minimal primary decomposition of L_G is

$$L_G =$$

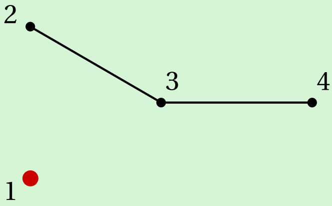
Example Let G be the graph



Then a minimal primary decomposition of L_G is

$$L_G = Q_{\varnothing}(G) =$$
$$(x_1x_2 + y_1y_2, x_1x_3 + y_1y_3, x_1x_4 + y_1y_4, x_2x_3 + y_2y_3, x_2x_4 + y_2y_4, x_3x_4 + y_3y_4,$$
$$x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_1y_4 - x_4y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3,$$
$$x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2, x_4^2 + y_4^2)$$

Example Let G be the graph

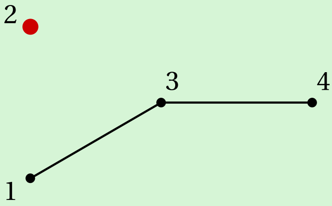


Then a minimal primary decomposition of L_G is

$$L_G = Q_{\emptyset}(G) \cap Q_{\{1\}}(G) =$$

$$(x_1x_2 + y_1y_2, x_1x_3 + y_1y_3, x_1x_4 + y_1y_4, x_2x_3 + y_2y_3, x_2x_4 + y_2y_4, x_3x_4 + y_3y_4,$$
$$x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_1y_4 - x_4y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3,$$
$$x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2, x_4^2 + y_4^2)$$
$$\cap (x_1, y_1, x_2x_3 + y_2y_3, x_3x_4 + y_3y_4, x_2y_4 - x_4y_2)$$

Example Let G be the graph



Then a minimal primary decomposition of L_G is

$$L_G = Q_{\varnothing}(G) \cap Q_{\{1\}}(G) \cap Q_{\{2\}}(G) =$$

$$(x_1x_2 + y_1y_2, x_1x_3 + y_1y_3, x_1x_4 + y_1y_4, x_2x_3 + y_2y_3, x_2x_4 + y_2y_4, x_3x_4 + y_3y_4,$$
$$x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_1y_4 - x_4y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3,$$
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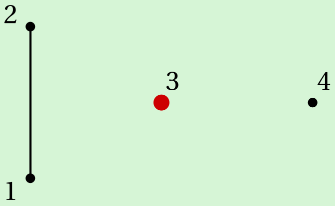
Example Let G be the graph



Then a minimal primary decomposition of L_G is

$$L_G = Q_{\varnothing}(G) \cap Q_{\{1\}}(G) \cap Q_{\{2\}}(G) \cap Q_{\{3\}}(G) =$$

$$(x_1x_2 + y_1y_2, x_1x_3 + y_1y_3, x_1x_4 + y_1y_4, x_2x_3 + y_2y_3, x_2x_4 + y_2y_4, x_3x_4 + y_3y_4,$$

$$x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_1y_4 - x_4y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3,$$

$$x_1^2 + y_1^2, x_2^2 + y_2^2, x_3^2 + y_3^2, x_4^2 + y_4^2)$$

$$\cap\, (x_1, y_1, x_2x_3 + y_2y_3, x_3x_4 + y_3y_4, x_2y_4 - x_4y_2)$$

$$\cap\, (x_2, y_2, x_1x_3 + y_1y_3, x_3x_4 + y_3y_4, x_1y_4 - x_4y_1)$$

$$\cap\, (x_3, y_3, x_1x_2 + y_1y_2).$$

Corollary *Let K be a field such that* $\mathrm{char}(K) \not\equiv 1, 2 \mod 4$ *or* $\mathrm{char}(K) = 0$. *Then the following are equivalent:*

- *the ideal $L_{\overline{G}}$ is prime,*
- *$\overline{G}$ is a disjoint union of edges and isolated vertices,*
- *G is $(n-2)$-connected.*

In this case, $L_{\overline{G}}$ is a complete intersection.

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Corollary *Let G be a graph with b bipartite connected components, and let $\sqrt{-1} \notin K$. Then L_G is unmixed if and only if $b(S) = |S| + b$ for every $\varnothing \neq S \in \mathcal{M}(G)$.*

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Future works

- *Which ideals L_G are Cohen-Macaulay (or Gorenstein, or complete intersection)? (jointly with Davide Bolognini)*
- *Can we say something about the resolution of L_G?*
- *Can we compute $\mathrm{pd}_T(T/L_G)$ and $\mathrm{reg}(L_G)$?*

Thank you for listening!