# $\mathbf{U}_q(\mathfrak{gl}_N)$ diagram categories via super $q$-Howe duality

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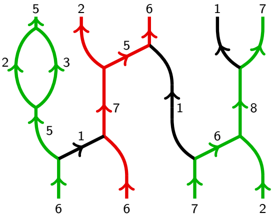
Or: the "diagrammatic presentation machine"




Joint work with David Rose, Pedro Vaz and Paul Wedrich

June 2015

1. Exterior $\mathfrak{gl}_N$-web categories
    - Graphical calculus via $N$-webs
    - Proof? Skew quantum Howe duality!

2. Exterior-symmetric $\mathfrak{gl}_N$-web categories
    - Its cousins: the green-red $N$-webs
    - Proof? Super quantum Howe duality!

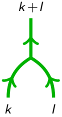
# History of diagrammatic presentations in a nutshell

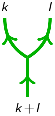
- Rumer, Teller, Weyl (1932):
  $\mathbf{U}(\mathfrak{sl}_2)$-tensor category generated by $\mathbb{C}^2$.
- Temperley-Lieb, Jones, Kauffman, Lickorish, Masbaum-Vogel ... ($\geq$1971):
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathbb{C}_q^2$.
- Kuperberg (1995):
  $\mathbf{U}_q(\mathfrak{sl}_3)$-tensor category generated by $\Lambda_q^1\mathbb{C}_q^3 \cong \mathbb{C}_q^3$ and $\Lambda_q^2\mathbb{C}_q^3$.
- Cautis-Kamnitzer-Morrison (2012):
  $\mathbf{U}_q(\mathfrak{sl}_N)$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^N$.
- Sartori (2013), Grant (2014):
  $\mathbf{U}_q(\mathfrak{gl}_{1|1})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{1|1}$.
- Rose-T. (2015):
  $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor category generated by $\mathrm{Sym}_q^k\mathbb{C}_q^2$.
- Link polynomials: Queffelec-Sartori (2015); "algebraic": Grant (2015):
  $\mathbf{U}_q(\mathfrak{gl}_{N|M})$-tensor category generated by $\Lambda_q^k\mathbb{C}_q^{N|M}$.

"Howe" do they fit in one framework?

# $\mathfrak{gl}_N$-webs

An *N-web* is an oriented, labeled, trivalent graph locally made of

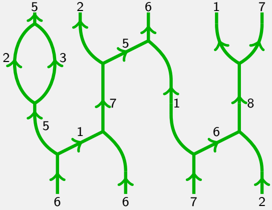
$\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{s}_{k+l}^{k,l} =$ 

$k, l, k+l \in \mathbb{N}$

(and no pivotal things today).

## Example



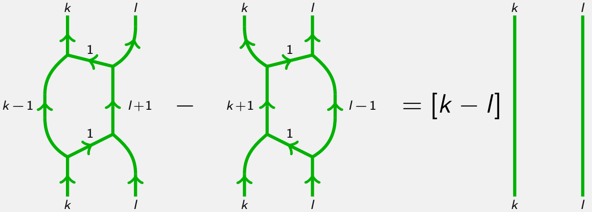
# Let us form a category

Define the (braided) monoidal, $\mathbb{C}_q$-linear category $N$-$\mathbf{Web}_{\mathfrak{g}}$ by using:

## Definition(Cautis-Kamnitzer-Morrison 2012)

The *$N$-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathfrak{g}}}(\vec{k},\vec{l})$ is the free $\mathbb{C}_q$-vector space generated by $N$-webs with $\vec{k}$ and $\vec{l}$ at the bottom and top modulo isotopies and:

$\mathfrak{gl}_m$ "ladder" relations :




Exterior relation :


$= 0$ , if $k > N$.

# Diagrams for intertwiners

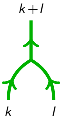
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_N)$-intertwiners

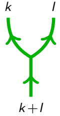
$$\mathrm{m}_{k,l}^{k+l}\colon \textstyle\bigwedge_q^k \mathbb{C}_q^N \otimes \bigwedge_q^l \mathbb{C}_q^N \twoheadrightarrow \bigwedge_q^{k+l}\mathbb{C}_q^N \quad \text{and} \quad \mathrm{s}_{k+l}^{k,l}\colon \bigwedge_q^{k+l}\mathbb{C}_q^N \hookrightarrow \bigwedge_q^k \mathbb{C}_q^N \otimes \bigwedge_q^l \mathbb{C}_q^N$$

given by projection and inclusion.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_e$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\bigwedge_q^k \mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$:

$$\vec{k} = (k_1, \dots, k_m) \mapsto \textstyle\bigwedge_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \bigwedge_q^{k_m}\mathbb{C}_q^N,$$



$\mapsto \mathrm{m}_{k,l}^{k+l}$ ,


$\mapsto \mathrm{s}_{k+l}^{k,l}$

**Theorem(Cautis-Kamnitzer-Morrison 2012)**

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{g}}^{\oplus} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$ is an equivalence of (braided) monoidal categories.

# “Howe” to prove this?

Howe: the commuting actions of $\mathbf{U}_q(\mathfrak{gl}_m)$ and $\mathbf{U}_q(\mathfrak{gl}_N)$ on

$$\Lambda_q^K(\mathbb{C}_q^m \otimes \mathbb{C}_q^N) \cong \bigoplus_{k_1+\cdots+k_m=K} (\Lambda_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \Lambda_q^{k_m}\mathbb{C}_q^N)$$

introduce an $\mathbf{U}_q(\mathfrak{gl}_m)$-action $f$ on the right term with $\vec{k}$-weight space $\Lambda_q^{\vec{k}}\mathbb{C}_q^N$.

In particular, there is a functor

$$\Phi_{\mathrm{skew}}^m \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e,$$
$$\vec{k} \mapsto \Lambda_q^{\vec{k}}\mathbb{C}_q^N, \quad X \in 1_{\vec{l}}\mathbf{U}_q(\mathfrak{gl}_m)1_{\vec{k}} \mapsto f(X) \in \mathrm{Hom}_{\mathfrak{gl}_N\text{-}\mathbf{Mod}_e}(\Lambda_q^{\vec{k}}\mathbb{C}_q^N, \Lambda_q^{\vec{l}}\mathbb{C}_q^N).$$

Howe: $\Phi_{\mathrm{skew}}^m$ is full. Or in words:

relations in $\dot{\mathbf{U}}_q(\mathfrak{gl}_m)$ + kernel of $\Phi_{\mathrm{skew}}^m \rightsquigarrow$ relations in $\mathfrak{gl}_N$-$\mathbf{Mod}_e$.

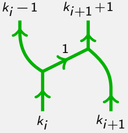
# Define the diagrams to make this work

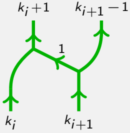
## Theorem(Cautis-Kamnitzer-Morrison 2012)

Define $N$-$\mathbf{Web}_{\mathrm{g}}$ such there is a commutative diagram

$$\dot{\mathbf{U}}_q(\mathfrak{gl}_m) \xrightarrow{\Phi^m_{\mathrm{skew}}} \mathfrak{gl}_N\text{-}\mathbf{Mod}_e, \qquad \Upsilon^m : \dot{\mathbf{U}}_q(\mathfrak{gl}_m) \to N\text{-}\mathbf{Web}_{\mathrm{g}}, \qquad \Gamma : N\text{-}\mathbf{Web}_{\mathrm{g}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_e$$

with

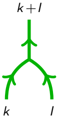
$\Upsilon^m(F_i 1_{\vec{k}}) \mapsto$ 

, $\Upsilon^m(E_i 1_{\vec{k}}) \mapsto$ 

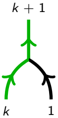

$\Upsilon^m \rightsquigarrow$ "$\mathfrak{gl}_m$ ladder" relations , $\ker(\Phi^m_{\mathrm{skew}}) \rightsquigarrow$ exterior relation.

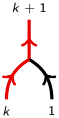
# Could there be a pattern?

A green-red $N$-web is a colored, labeled, trivalent graph locally made of

$\mathrm{m}_{k,l}^{k+l} =$ 

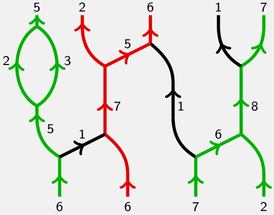
, $\mathrm{m}_{k,l}^{k+l} =$ 

, $\mathrm{m}_{k,1}^{k+l} =$ 

, $\mathrm{m}_{k,1}^{k+l} =$ 


And of course splits and some mirrors as well!

## Example



# The green-red $N$-web category

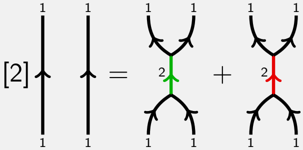
Define the (braided) monoidal, $\mathbb{C}_q$-linear category $N$-$\mathbf{Web}_{\mathrm{gr}}$ by using:

## Definition

Given $\vec{k} \in \mathbb{Z}_{\geq 0}^{m+n}, \vec{l} \in \mathbb{Z}_{\geq 0}^{m'+n'}$. The green-red *$N$-web space* $\mathrm{Hom}_{N\text{-}\mathbf{Web}_{\mathrm{gr}}}(\vec{k}, \vec{l})$ is the free $\mathbb{C}_q$-vector space generated by $N$-webs between $\vec{k}$ and $\vec{l}$ modulo isotopies and:

$\mathfrak{gl}_m + \mathfrak{gl}_n$ "ladder" relations : same as before, but now in red as well!

Dumbbell relation :




Exterior relation :


$= 0$ , if $k > N$.

# Diagrams for intertwiners - Part 2

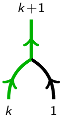
Observe that there are (up to scalars) unique $\mathbf{U}_q(\mathfrak{gl}_N)$-intertwiners

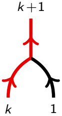
$$\mathrm{m}_{k,1}^{k+1}\colon \Lambda_q^k\mathbb{C}_q^N \otimes \mathbb{C}_q^N \twoheadrightarrow \Lambda_q^{k+1}\mathbb{C}_q^N \quad \text{and} \quad \mathrm{m}_{k,1}^{k+1}\colon \mathrm{Sym}_q^k\mathbb{C}_q^N \otimes \mathbb{C}_q^N \twoheadrightarrow \mathrm{Sym}_q^{k+1}\mathbb{C}_q^N$$

plus others as before.

Let $\mathfrak{gl}_N$-$\mathbf{Mod}_{\mathrm{es}}$ be the (braided) monoidal, $\mathbb{C}_q$-linear category whose objects are tensor generated by $\Lambda_q^k\mathbb{C}_q^N, \mathrm{Sym}_q^k\mathbb{C}_q^N$. Define a functor $\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$:

$$\vec{k} = (k_1, \dots, k_m, k_{m+1}, \dots, k_{m+n}) \mapsto \Lambda_q^{k_1}\mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}}\mathbb{C}_q^N,$$



$\mapsto \mathrm{m}_{k,1}^{k+1}$ ,


$\mapsto \mathrm{m}_{k,1}^{k+1}$ , $\cdots$

## Theorem

$\Gamma\colon N\text{-}\mathbf{Web}_{\mathrm{gr}}^{\oplus} \to \mathfrak{gl}_N\text{-}\mathbf{Mod}_{\mathrm{es}}$ is an equivalence of (braided) monoidal categories.

# Super $\mathfrak{gl}_{m|n}$

## Definition

The *quantum general linear superalgebra* $\mathbf{U}_q(\mathfrak{gl}_{m|n})$ is generated by $L_i^{\pm 1}$ and $F_i, E_i$ subject the some relations, most notably, the *super relations*:

$$F_m^2 = 0 = E_m^2 \quad , \quad \frac{L_m L_{m+1}^{-1} - L_m^{-1} L_{m+1}}{q - q^{-1}} = F_m E_m + E_m F_m,$$

$$\begin{aligned}[2] F_m F_{m+1} F_{m-1} F_m =& F_m F_{m+1} F_m F_{m-1} + F_{m-1} F_m F_{m+1} F_m \\ &+ F_{m+1} F_m F_{m-1} F_m + F_m F_{m-1} F_m F_{m+1} \text{ (plus an E version).}\end{aligned}$$

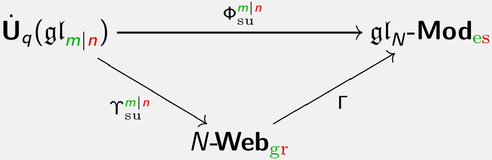
There is a Howe pair $(\mathbf{U}_q(\mathfrak{gl}_{m|n}), \mathbf{U}_q(\mathfrak{gl}_N))$ with $\vec{k} = (k_1, \dots, k_{m+n})$-weight space under the $\mathbf{U}_q(\mathfrak{gl}_{m|n})$-action on $\Lambda_q^K(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^N)$ given by

$$\Lambda_q^{k_1} \mathbb{C}_q^N \otimes \cdots \Lambda_q^{k_m} \mathbb{C}_q^N \otimes \mathrm{Sym}_q^{k_{m+1}} \mathbb{C}_q^N \otimes \cdots \otimes \mathrm{Sym}_q^{k_{m+n}} \mathbb{C}_q^N.$$

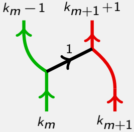
# Define the diagrams to make this work

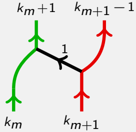
## Theorem

Define $N$-$\mathbf{Web}_{\mathrm{gr}}$ such there is a commutative diagram




with

$\Upsilon^{m|n}_{\mathrm{su}}(F_m 1_{\vec{k}}) \mapsto$ 


, $\Upsilon^{m|n}_{\mathrm{su}}(E_m 1_{\vec{k}}) \mapsto$ 


$\Upsilon^{m|n}_{\mathrm{su}} \rightsquigarrow$ "$\mathfrak{gl}_{m|n}$ ladder" relations , $\ker(\Phi^{m|n}_{\mathrm{su}}) \rightsquigarrow$ the exterior relation.

# Some concluding remarks

- Feed the machine with the *Howe pair* $(\mathfrak{gl}_{m|n}, \mathfrak{gl}_{N|M})$ and one gets a diagrammatic presentation of $\mathfrak{gl}_{N|M}$-$\mathbf{Mod}_{\mathrm{es}}$.
- Taking $N, M \to \infty$, one obtains a diagrammatic presentation $\infty$-$\mathbf{Web}_{\mathrm{gr}}$ of some form of the Hecke algebroid. Roughly: the machine spits it out, if you feed it with *Schur-Weyl duality*. This also gives a new presentation of the super $q$-Schur algebra - without the fancy super relations.
- $\infty$-$\mathbf{Web}_{\mathrm{gr}}$ is completely symmetric in green-red which allows us to prove a symmetry of HOMFLY-PT polynomials diagrammatically:

$$\mathcal{P}^{a,q}(\mathcal{L}(\vec{\lambda})) = (-1)^{co}\, \mathcal{P}^{a,q^{-1}}(\mathcal{L}(\vec{\lambda}^{\mathrm{T}})).$$

- Homework: feed the machine with your favorite duality (e.g. Howe dualities in other types) and see what it spits out.
- Everything is (hopefully) amenable to categorification!

There is still much to do...

Thanks for your attention!