An Arrow-like theorem for aggregation procedures over median algebras

Miguel Couceiro

LORIA

(CNRS, INRIA Nancy G.–E., Université de Lorraine)

Joint work with J-L Marichal and B. Teheux (Luxembourg), G. Meletiou (Greece)

Outline

I. Brief overview on aggregation theory :

I.1. Aggregation functions : motivation

I.2. An impossibility result : Arrow's theorem

II. Aggregation over median algebras :

II.1. Median algebras : motivation and examples

II.2. Conservative median algebras

II.3. Median preserving aggregation : An Arrow-like theorem

II.4. Final remarks and further directions...

I.1 Aggregation functions

Traditionally : an **aggregation function** is a mapping $F\colon X^n \to X$ **s.t.**

- X is a linear order with bottom 0 and top 1
- F preserves 0 and 1 and the order of X

Typical examples : Weighted means, Choquet and Sugeno integrals . . .

Main Idea : Aggregation procedure $x_1, \ldots, x_n \in X \quad \longrightarrow \quad F(x_1, \ldots, x_n) \in Y$

Application : Preference modelling (MCDA) . . .

Main Problems :

- Classify and axiomatise aggregation procedures
- *Explicitly describe procedures with desired properties*
- Computational aspects

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I.2. An impossibility result : Arrow's theorem

Setting : Aggregation of rankings (social well-fare function)

- n voters, a set A of outcomes and the set of linear orderings $L(A)$
- $F\colon L(A)^n \to L(A)$ procedure that merges rankings $R_1, \ldots, R_n$ into a single one

$$R_1, \ldots, R_n \quad \Longrightarrow \quad R_T = F(R_1, \ldots, R_n)$$

- Some reasonable properties in this setting :

1. **Unanimity or Pareto efficiency :** if aR_ib for all $i \in [n]$, **then** aR_Tb
2. **Independence of irrelevant alternatives :** if a and b have the same order in R_i and S_i for all $i \in [n]$, **then** a and b have the same order in R_T and S_T
3. **Non-dictatorship :** There is no $i \in [n]$ **s.t.** for all $R_1, \ldots, R_n \in L(A)$

$$F(R_1, \ldots, R_n) = R_i$$

Arrow's Theorem : There is no well-fare function satisfing these conditions !

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II.1. Median algebras : motivation

Median operations appear in several structures pertaining to decision making :

- **Linear orders :** "in betweeness"
- **Distributive lattices :** $\mathbf{m}(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

Theorem : A function $f\colon X^n \to X$ is a lattice polynomial function **iff**

$$f(\mathbf{x}) = \mathbf{m}\big(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1)\big) \quad \text{for every } \mathbf{x} \in X^n,\, k \in [n]$$

Median algebra : Structure $\mathbf{A} = (A, \mathbf{m})$ where $\mathbf{m}\colon A^3 \to A$ (median) verifies

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II.1. Median algebras : equivalent structures

Other known median algebras :

- **Median semilattices :** For $a \in A$, $\downarrow a$ is a DBLattice **and** every $x, y, z \in A$ have a common upper bound **whenever** each pair of them is bounded above.

  NB1 : If A median algebra, set $x \leq_a y \iff \mathbf{m}(a, x, y) = x$

  NB2 : If A median semilattice, set $\mathbf{m}(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

- **Median graphs :** For all x, y, z, there is a unique w in the shortest paths

  NB1 : Every median semilattice (with finite intervals) has a median Hasse diag.

  NB2 : Every median graph is the Hasse diagram of a median semilattice

References : Barthélemy-Leclerc-Monjardet'86, Bandelt'83, Isbell'80, Avann'61, ...

Generalisations : Bandelt-Meletiou'92, Barthélemy-Janowitz'91, Bandelt'90, ...

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II.2. Conservative median algebras

Conservative median algebra : If $\mathbf{m}(x,y,z) \in \{x,y,z\}, \qquad x,y,z \in A$

Social choice motivation : the median candidate is one of the candidates

Problem : How do they look like ?

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Representation of conservative median algebras

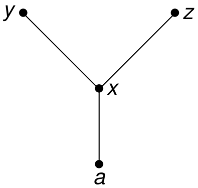
Theorem : Let $\mathbf{A}$ be a median algebra with $|A| \geq 5$. T.F.A.E.

(i) $\mathbf{A}$ is conservative.

(ii) There is an $a \in A$ and lower bounded chains $\mathbf{C}_0$ and $\mathbf{C}_1$ such that $\langle A, \leq_a \rangle$ is isomorphic to $\mathbf{C}_0 \perp \mathbf{C}_1$.

(iii) For every $a \in A$, there are lower bounded chains $\mathbf{C}_0$ and $\mathbf{C}_1$ such that $\langle A, \leq_a \rangle$ is isomorphic to $\mathbf{C}_0 \perp \mathbf{C}_1$.

(iv) For every $a \in A$ the ordered set $\langle A, \leq_a \rangle$ does not contain a copy of the poset



Open problem : Representation of arbitrary median algebras

II.3. Median preserving aggregation

Idea : Score of a median profile is the median of the scores of the profiles

Problem : Aggregation functions $f\colon X^n \to Y$ that preserve medians :

$$f(\mathbf{m}(\mathbf{x},\mathbf{y},\mathbf{z})) \;=\; \mathbf{m}(f(\mathbf{x}),f(\mathbf{y}),f(\mathbf{z})),$$

Remark : median preserving maps are not necessarily order-preserving (reversing) !

An order-preserving map that is not median preserving

A median preserving map that is not order-preserving (or reversing)

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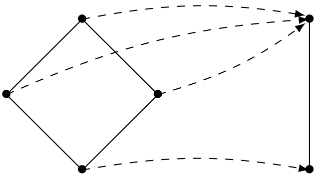
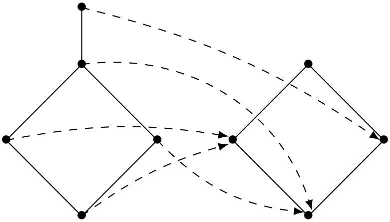


An order-preserving map that is not median preserving



A median preserving map that is not order-preserving (or reversing)

Characterization of median preserving maps

NB : Every conservative median algebra $\mathbf{A}$ can be thought of as a chain $\mathbf{C}(\mathbf{A})$

Theorem : Let $\mathbf{A}, \mathbf{B}$ be conservative median algebras with ≥ 5 elements. T.F.A.E. :

(i) $f : \mathbf{A} \to \mathbf{B}$ is a median preserving map

(ii) the induced map $f' : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(\mathbf{B})$ is order-preserving or order-reversing

Problem : How to lift it to $f : \mathbf{A}^n \to \mathbf{B}$

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Back to aggregation functions...

Theorem : Let $\mathbf{A} = \mathbf{C}_1 \times \cdots \times \mathbf{C}_n$ and $\mathbf{B} = \mathbf{D}_1 \times \cdots \times \mathbf{D}_k$ be products of chains. T.F.A.E. :

(i) $f : \mathbf{A} \to \mathbf{B}$ is median preserving

(ii) there exist $\sigma : [k] \to [n]$ and order-preserving or order-reversing maps

$$f_i : \mathbf{C}_{\sigma(i)} \to \mathbf{D}_i \quad \text{for } i \in [k] \quad \textbf{s.t.} \; f(\mathbf{x}) = (f_1(x_{\sigma(1)}), \ldots, f_k(x_{\sigma(k)}))$$

Corollary : Let $\mathbf{C}_1, \cdots, \mathbf{C}_n$ and $\mathbf{D}$ (in part., $k = 1$) be chains. T.F.A.E. :

(i) $f : \mathbf{C}_1 \times \cdots \times \mathbf{C}_n \to \mathbf{D}$ is median preserving

(ii) there is $j \in [n]$ and order-preserving or reversing map $g : \mathbf{C}_j \to \mathbf{D}$ **s.t.** $f = g \circ \pi_j$

Consequence : Arrow-like theorem over median algebras

Aggregation procedures that preserve medians are dictatorial !

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Obrigado pela vossa atenção !