Poisson deleting derivations algorithm: Poisson birational equivalence and Poisson spectrum

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Poisson Algebra

Definition

A Poisson $\mathbb{K}$*-algebra A is a commutative algebra endowed with a Poisson bracket, that is a skew-symmetric* $\mathbb{K}$*-bilinear map from* $A \times A$ *to A such that for all* $a, b, c \in A$ *we have:*

- $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0,$
- $\{ab, c\} = a\{b, c\} + \{a, c\}b.$

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Example:

Let $A := \mathcal{O}\big(M_2(\mathbb{K})\big) = \mathbb{K}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$. We define a Poisson structure on A by setting:

$$\begin{aligned} &\{X_{11}, X_{12}\} = X_{11}X_{12}, \quad \{X_{11}, X_{22}\} = 2X_{12}X_{21}, \quad \{X_{12}, X_{22}\} = X_{12}X_{22}, \\ &\{X_{11}, X_{21}\} = X_{11}X_{21}, \quad \{X_{12}, X_{21}\} = 0, \quad \{X_{21}, X_{22}\} = X_{21}X_{22}. \end{aligned}$$

We generalise this Poisson structure to $\mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$.

Poisson-Ore extension $A \hookrightarrow A[X]$

Let A be a Poisson algebra and α, δ be linear maps on A. By setting:

$$\{X, a\} = \alpha(a)X + \delta(a) \text{ for all } a \in A,$$

we define a Poisson bracket on the polynomial ring $A[X]$ if and only if α, δ are derivations of A and satisfies for all $a, b \in A$:

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\},$$
$$\delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\} + \alpha(a)\delta(b) - \delta(a)\alpha(b).$$

We denote this Poisson algebra by $A[X; \alpha, \delta]_P$ and called it a Poisson-Ore extension.

Poisson Deleting Derivation Homomorphism

Theorem (Launois-L.)

Let $A[X; \alpha, \delta]_P$ be a Poisson-Ore extension and $\eta \in \mathbb{K}^\times$. Suppose that δ extends to an iterative, locally nilpotent higher (η, α)-skew Poisson derivation (D_i) on A.

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$$F : A[Y^{\pm 1};\alpha,0]_P \xrightarrow{\cong} A[X^{\pm 1};\alpha,\delta]_P,$$

defined by

$$F(a) = \sum_{i \geq 0} \frac{1}{\eta^i} D_i(a) X^{-i}, \quad \text{and} \quad F(Y) = X.$$

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Compare $\{Y,a\} = \alpha(a)Y$ and $\{X,a\} = \alpha(a)X + \delta(a)$.

Higher Poisson Derivation

Let A be a Poisson $\mathbb{K}$-algebra, α a Poisson derivation, and $\eta \in \mathbb{K}$.

A higher (η, α)-skew Poisson derivation is a sequence of $\mathbb{K}$-linear maps $(D_i) := (D_i)_{i=0}^{\infty}$ from A to A such that for all $a, b \in A$ and all $n \geq 0$:

(1) $D_0 = \mathrm{id}_A$ and $D_n(ab) = \sum_{i=0}^{n} D_i(a)D_{n-i}(b)$,

(2) $D_n(\{a, b\}) =$
$$\sum_{i=0}^{n} \Big[\{D_i(a), D_{n-i}(b)\} + i\big(\alpha D_{n-i}(a)D_i(b) - D_i(a)\alpha D_{n-i}(b)\big)\Big],$$

(3) $D_n\alpha = \alpha D_n + n\eta D_n$,

(4) $D_i D_j = \binom{i+j}{i} D_{i+j}$ for all $i, j \geq 0$ (iterative),

(5) For all $a \in A$ there exists $n_a \geq 0$ such that $D_i(a) = 0$ for all $i \geq n_a$ (locally nilpotent).

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If $\operatorname{char} \mathbb{K} = 0$, then $D_i = \frac{D_1^i}{i!}$ for all i.

A class $\mathcal{P}$ of Poisson algebras

- $A = \mathbb{K}[X_1][X_2; \alpha_2, \delta_2]_P \cdots [X_n; \alpha_n, \delta_n]_P$ is an iterated Poisson-Ore extension.
- $\alpha_i(X_j) = \lambda_{ij} X_j$ for some scalar $\lambda_{ij} \in \mathbb{K}$. $(1 \leq j < i \leq n)$
- δ_i extends to a higher (η_i, α_i)-skew Poisson derivation $(D_{i,k})_{k=0}^{\infty}$, where $\eta_i \in \mathbb{K}^{\times}$. $(2 \leq i \leq n)$
- $\alpha_i D_{j,k} = D_{j,k} \alpha_i + k \lambda_{ij} D_{j,k}$ for all $k \geq 0$. $(2 \leq j < i \leq n)$

The class $\mathcal{P}$ contains (the coordinate rings of) Poisson matrix variety $\mathcal{O}\big(M_{m,p}(\mathbb{K})\big)$, odd and even Poisson euclidean spaces and symplectic Poisson spaces.

Consequences

Let $A \in \mathcal{P}$. Then the Poisson deleting derivations algorithm returns a Poisson affine space $\overline{A} = \mathbb{K}[T_1, \dots, T_n]$ with $\{T_i, T_j\} = \lambda_{ij} T_i T_j$ such that:

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Corollary

Let $A \in \mathcal{P}$. Then A satisfies the Poisson analogue of the quantum Gel'fand-Kirillov problem. Moreover, if $\mathrm{char}\,\mathbb{K} = 0$, then A satisfies the Poisson Dixmier-Moeglin equivalence.

Canonical partition and Cauchon Diagrams

Let $A \in \mathcal{P}$, and $\overline{A} = \mathbb{K}[T_1, \ldots, T_n]$ with $\{T_i, T_j\} = \lambda_{ij} T_i T_j$. Let $W := \mathscr{P}(\{1, \ldots, n\})$.

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$$\mathrm{P.Spec}_w(\overline{A}) := \left\{ P \in \mathrm{P.Spec}\,(\overline{A}) \mid P \cap \{T_1, \ldots, T_n\} = \{T_i \mid i \in w\} \right\}.$$

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induces a partition of $\mathrm{P.Spec}(A)$:

$$\mathrm{P.Spec}(A) = \bigsqcup_{w \in W'_P} \varphi^{-1}(\mathrm{P.Spec}_w(\overline{A}))$$

where $W'_P := \{w \in W \mid \varphi^{-1}(\mathrm{P.Spec}_w(\overline{A})) \neq \emptyset\}$ (the set of **Cauchon diagrams**).

Combinatorial description of Cauchon diagram for Poisson matrices $A = \mathcal{O}(M_{m,p}(\mathbb{K}))$

In the case we have:

$$W := \mathscr{P}(\{1, \ldots, m\} \times \{1, \ldots, p\}),$$

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Theorem

When $\operatorname{char} \mathbb{K} \neq 2$, the set of Cauchon diagrams W'_P for A is in bijection with the set of all $m \times p$ rectangular arrays whose boxes are colored in black or white with the property that if a box is black, then all the boxes above it are black or all the boxes on its left are black.