Heteroclinic cycles and networks in coupled cell systems

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includes joint-work with
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Sofia castro (University of Porto) **Mike Field** (Rice University)
Haibo Ruan (Universität Hamburg)

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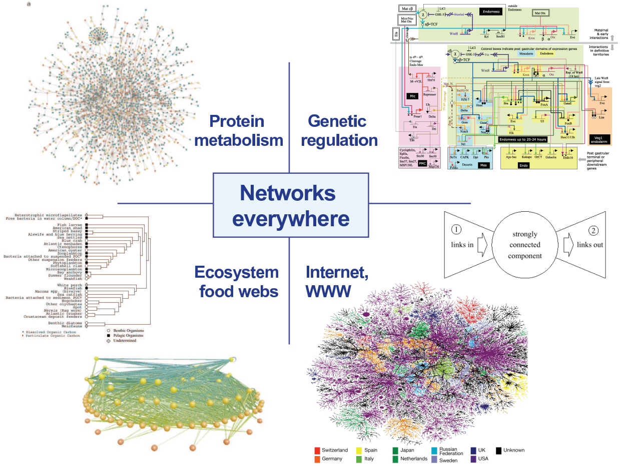
Coupled cell networks are everywhere!



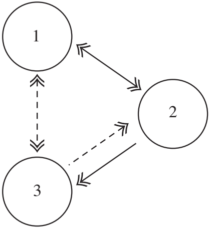
http://perso.uclouvain.be/vincent.blondel/workshops/2008/files/barahona.pdf

Heteroclinic phenomena in coupled cell networks

In terms of applications, particularly in computational neuroscience, heteroclinic phenomena in coupled cell networks have been attracting a growing interest.

Coupled cell network (CCN) and coupled cell system (CCS)

CCNs realised as CCSs modelled by ordinary differential equations.



$$\begin{array}{l} \dot{x}_1 = f(x_1; x_2, x_3) \\ \dot{x}_2 = f(x_2; x_1, x_3) \\ \dot{x}_3 = f(x_3; x_2, x_1) \end{array} ,$$

with $f: \mathbf{R}^3 \to \mathbf{R}$ a smooth function.

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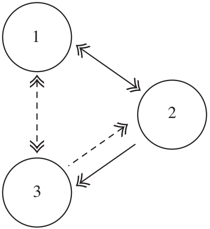

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with $f : \mathbf{R}^3 \to \mathbf{R}$ a smooth function.

Here, we consider:

- identical cells
- one-dimensional cell dynamics

Synchrony subspace

A *synchrony subspace* associated with a CCN is a subspace of the total phase space

- defined by equality of cell coordinates
- invariant by the flow of all the CCS associated with the CCN.

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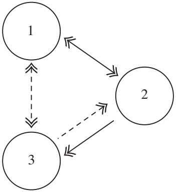

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_2, x_3) \\ \dot{x}_2 &= f(x_2; x_1, x_3) \\ \dot{x}_3 &= f(x_3; x_2, x_1) \end{aligned}$$

$$\{\mathbf{x} : x_1 = x_2\}$$
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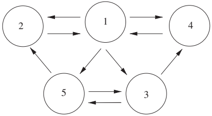
Lattice of synchrony subspaces

- For identical cell CCNs the maximal synchrony subspace is the full synchrony subspace.
- The set of synchrony subspaces of a CCN is a complete lattice with partial order relation given by inclusion.
- In [AD14], we show how to obtain the lattice of synchrony subspaces for a general network and present an algorithm that generates that lattice.

[AD14] A. and Dias, *The Lattice of Synchrony Subspaces of a Coupled Cell Network: Characterization and Computation Algorithm*, Journal of Nonlinear Science 24 (6) (2014) 949-996.

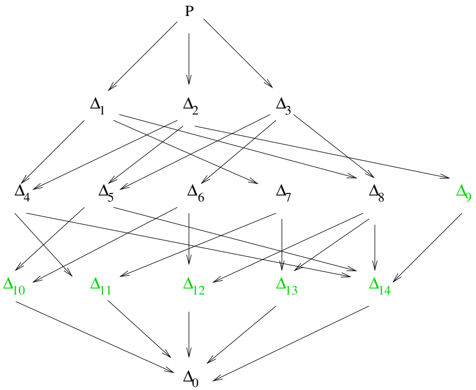
Regular network with semi-simple adjacency matrix



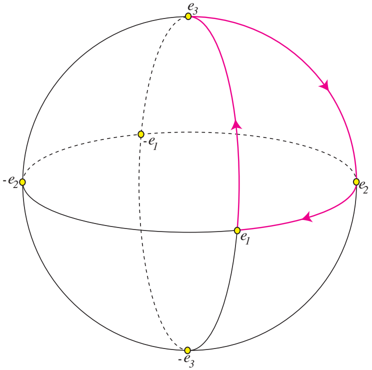
$$
\begin{aligned}
\dot{x}_1 &= f(x_1, \overline{x_2, x_4}) \\
\dot{x}_2 &= f(x_2, \overline{x_1, x_5}) \\
\dot{x}_3 &= f(x_3, \overline{x_1, x_5}) \\
\dot{x}_4 &= f(x_4, \overline{x_1, x_3}) \\
\dot{x}_5 &= f(x_5, \overline{x_1, x_3})
\end{aligned}
$$

$$
\begin{aligned}
\Delta_1 &= \{x : \; x_2 = x_3\} \\
\Delta_2 &= \{x : \; x_3 = x_5\} \\
\Delta_3 &= \{x : \; x_4 = x_5\} \\
\\
\Delta_4 &= \{x : \; x_2 = x_3 = x_5\} \\
\Delta_5 &= \{x : \; x_3 = x_4 = x_5\} \\
\Delta_6 &= \{x : \; x_1 = x_2, \; x_4 = x_5\} \\
\Delta_7 &= \{x : \; x_1 = x_4, \; x_2 = x_3\} \\
\Delta_8 &= \{x : \; x_2 = x_3, \; x_4 = x_5\} \\
\Delta_9 &= \{x : \; x_2 = x_4, \; x_3 = x_5\} \\
\\
\Delta_{10} &= \{x : \; x_1 = x_2, \; x_3 = x_4 = x_5\} \\
\Delta_{11} &= \{x : \; x_1 = x_4, \; x_2 = x_3 = x_5\} \\
\Delta_{12} &= \{x : \; x_1 = x_2 = x_3, \; x_4 = x_5\} \\
\Delta_{13} &= \{x : \; x_1 = x_4 = x_5, \; x_2 = x_3\} \\
\Delta_{14} &= \{x : \; x_2 = x_3 = x_4 = x_5\}
\end{aligned}
$$

The lattice



Heteroclinic cycle and network



Heteroclinic cycle : cycle of invariant saddle equilibria connected by trajectories (corresponding to nontrivial intersections of their invariant manifolds).

Heteroclinic network : connected finite union of heteroclinic cycles.

Robust simple heteroclinic cycle

In [AADF11], we are interested in:

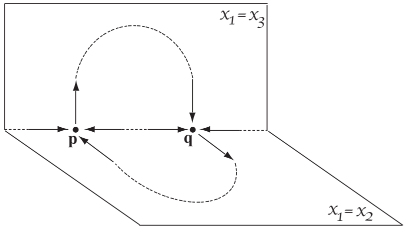
- *robust heteroclinic cycles* contained in flow invariant subspaces forced by the network structure (synchrony subspaces) but not given by symmetry;
- *simple heteroclinic cycles*, i.e., the unstable manifold of each saddle point is one dimensional and contained in the stable manifold of another saddle point lying on an invariant subspace;
- *attracting heteroclinic cycles*, in the sense that an open set in the neighbourhood of the cycle is attracted to the cycle.

[AADF11] A., Ashwin, Dias, and Field, *Dynamics of coupled cell networks: Synchrony, heteroclinic cycles and inflation*, Journal of Nonlinear Science, 21(2) (2011), 271-323.

Necessary condition

A necessary condition for the existence of robust simple heteroclinic cycles is the

- existence of two non-maximal synchrony subspaces.



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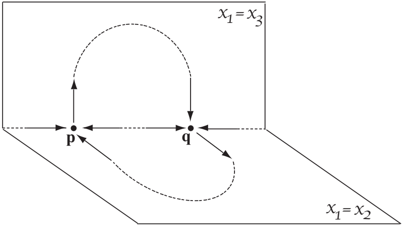
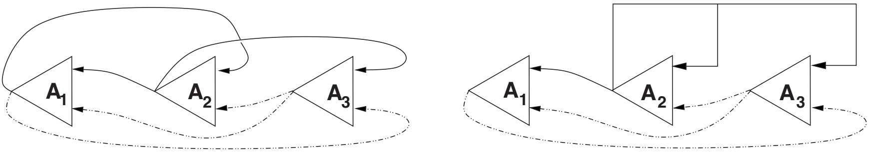


This excludes 2-cell networks and single input cell networks.

Three-cell networks

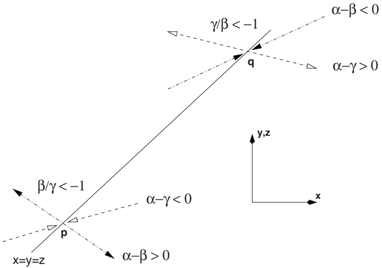
We consider 3-cell networks with two asymmetric inputs.



There are two equivalence classes of three identical cell networks that admit robust simple heteroclinic cycles.

Local issue

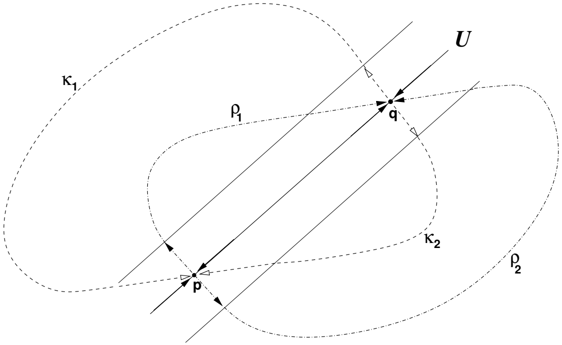
Show that there are vector fields consistent with the network structure with two saddle equilibrium points in the maximal synchrony subspace having a one dimensional unstable manifold lying in one of the two dimensional synchrony subspaces.



(Synchrony subspaces $x_1 = x_2$ and $x_1 = x_3$ are projected onto the same plane.)

Global issue

Show that we can extend those vector fields so that we obtain saddle connections between the two totally synchronous equilibria defining a simple heteroclinic cycle.



Recent work

More recently:

- Ashwin and Postlethwaite [AP13] presented two methods of realising arbitrarily complex directed graphs as robust heteroclinic networks for flows generated by coupled ordinary differential equations,
- Field [F15], proved results that enable the realisation of heteroclinic networks in coupled homogeneous and heterogeneous systems of identical cells.

[AP13] Ashwin and Postlethwaite, *On designing heteroclinic networks from graphs*, Phys. D 265 (2013) 2639.

[F15] Field, *Heteroclinic networks in homogeneous and heterogeneous identical cell systems,* Journal of Nonlinear Science, 25(3) (2015), 779-813.

Motivation

The description of heteroclinic dynamical behaviour

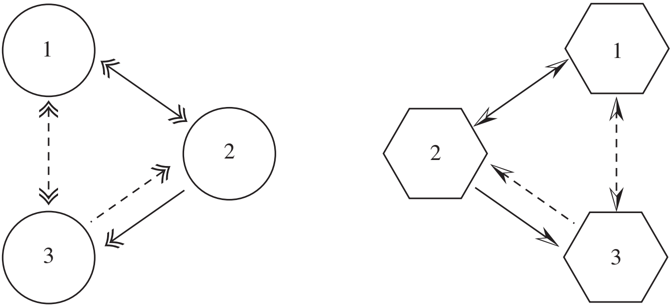
- in higher dimensional CCS associated with a larger CCN
- based on the heteroclinic dynamics of the CCS consistent with the structure of smaller component networks.

We consider:

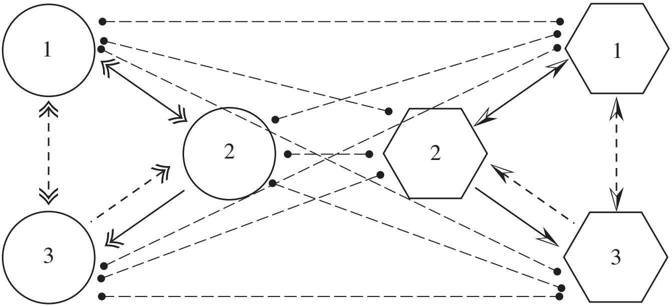
- join networks,
- product networks.

[AAD] A., Ashwin and Dias, *Heteroclinic networks dynamics on joining coupled cell networks*, in preparation.

Join of networks



Join of networks



Heteroclinic networks in the join of networks

In [AAD], assuming that

- the CCS for $\mathcal{N}_1$ support a robust simple heteroclinic cycle involving the totally synchronous equilibria $\mathbf{p} = (p, p, p)$ and $\mathbf{q} = (q, q, q)$
- the CCS for $\mathcal{N}_2$ support a robust simple heteroclinic cycle involving the totally synchronous equilibria $\overline{\mathbf{p}} = (\overline{p}, \overline{p}, \overline{p})$ and $\overline{\mathbf{q}} = (\overline{q}, \overline{q}, \overline{q})$

we prove that

- there are CCS for the join network $\mathcal{N}_1 * \mathcal{N}_2$ that support the existence of a robust heteroclinic network involving the partially synchronous equilibria $(\mathbf{p}, \overline{\mathbf{p}})$, $(\mathbf{p}, \overline{\mathbf{q}})$, $(\mathbf{q}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$.

[AAD] A., Ashwin and Dias, *Heteroclinic networks dynamics on joining coupled cell networks*, in preparation.

Synchrony subspaces for the join of networks

In [AR12], we obtain classification results for synchrony subspaces of the join network, which give

- the relation between the lattice of synchrony subspaces of the join network and its components.

In this case, since we are assuming the cell type of the cells in the two networks are different,

- the synchrony subspace for the join network are the ones given by the product of the synchrony subspaces of the component networks.

[AR12] A. and Ruan, *Evolution of Synchrony under Combination of Coupled Cell Networks*, Nonlinearity 25 (2012) 3155-3187.

Synchrony subspaces for the join network

$$
\begin{array}{l}
P_1 \times P_2 \\
P_1 \times S_2^2 = \{(\mathbf{x},\mathbf{y}) : \ y_1 = y_2\} \\
P_1 \times S_2^3 = \{(\mathbf{x},\mathbf{y}) : \ y_1 = y_3\} \\
P_1 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}) : \ y_1 = y_2 = y_3\} \\
\hline
S_1^2 \times P_2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2\} \\
S_1^2 \times S_2^2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2, \ y_1 = y_2\} \\
S_1^2 \times S_2^3 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2, \ y_1 = y_3\} \\
S_1^2 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2, \ y_1 = y_2 = y_3\} \\
\hline
S_1^3 \times P_2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_3\} \\
S_1^3 \times S_2^2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_3, \ y_1 = y_2\} \\
S_1^3 \times S_2^3 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_3, \ y_1 = y_3\} \\
S_1^3 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_3, \ y_1 = y_2 = y_3\} \\
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\Delta_1 \times P_2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2 = x_3\} \\
\Delta_1 \times S_2^2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2 = x_3, \ y_1 = y_2\} \\
\Delta_1 \times S_2^3 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2 = x_3, \ y_1 = y_3\} \\
\Delta_1 \times \Delta_2 = \{(\mathbf{x},\mathbf{y}) : \ x_1 = x_2 = x_3, \ y_1 = y_2 = y_3\}
\end{array}
$$

Admissible CCS for the join network

We consider admissible CCS for the join network of the form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1; x_2, x_3)\,[1 + h(\overline{y_1, y_2, y_3})]\\
\dot{x}_2 &= f_1(x_2; x_1, x_3)\,[1 + h(\overline{y_1, y_2, y_3})]\\
\dot{x}_3 &= f_1(x_3; x_2, x_1)\,[1 + h(\overline{y_1, y_2, y_3})]\\
\dot{y}_1 &= f_2(y_1; y_2, y_3)\,[1 + h(\overline{x_1, x_2, x_3})]\\
\dot{y}_2 &= f_2(y_2; y_1, y_3)\,[1 + h(\overline{x_1, x_2, x_3})]\\
\dot{y}_3 &= f_2(y_3; y_2, y_1)\,[1 + h(\overline{x_1, x_2, x_3})]
\end{aligned}$$

Trivially, $(\mathbf{p}, \overline{\mathbf{p}})$, $(\mathbf{p}, \overline{\mathbf{q}})$, $(\mathbf{q}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$ are equilibrium points for the join systems.

Stability of equilibria

We show that the Jacobian $J(\mathbf{x}, \mathbf{y})$ is similar to

$$\mathrm{diag}\left((1 + h(\mathbf{y}))J(\mathbf{x}), (1 + h(\mathbf{x}))J(\mathbf{y})\right).$$

Thus, if

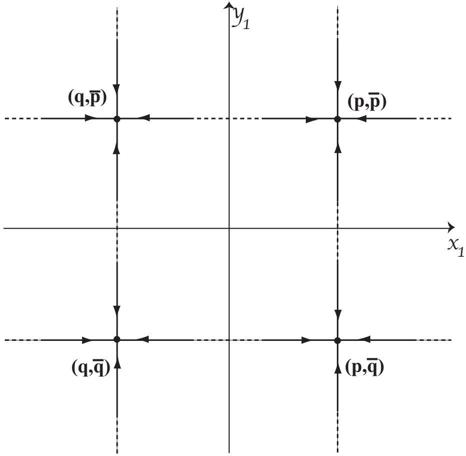
$$1 + h(\overline{\mathbf{p}}) > 0, \qquad 1 + h(\overline{\mathbf{q}}) > 0,$$

$$1 + h(\mathbf{p}) > 0, \qquad 1 + h(\mathbf{q}) > 0\,,$$

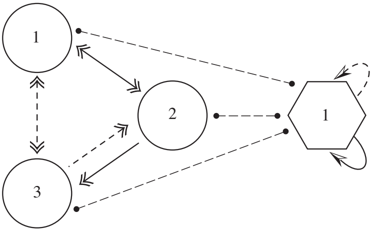
then for each such equilibrium $(\mathbf{x}, \mathbf{y})$,

- the signs of the six eigenvalues of the Jacobian $J(\mathbf{x}, \mathbf{y})$ correspond
  - to the signs of the three eigenvalues of the Jacobian $J(\mathbf{x})$ and
  - of the three eigenvalues of the Jacobian $J(\mathbf{y})$.

Stability in $\Delta_1 \times \Delta_2 = \{(\mathbf{x}, \mathbf{y}) : \; x_1 = x_2 = x_3, \; y_1 = y_2 = y_3\}$



Quotient network in the restriction to $P_1 \times \Delta_2 = \{(\mathbf{x}, \mathbf{y}) : \; y_1 = y_2 = y_3\}$



For $\mathbf{y} = (\overline{\overline{p}, \overline{p}, \overline{p}})$ we have

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1; x_2, x_3)\left[1 + h(\overline{\overline{p}, \overline{p}, \overline{p}})\right] \\
\dot{x}_2 &= f_1(x_2; x_1, x_3)\left[1 + h(\overline{\overline{p}, \overline{p}, \overline{p}})\right] \\
\dot{x}_3 &= f_1(x_3; x_2, x_1)\left[1 + h(\overline{\overline{p}, \overline{p}, \overline{p}})\right] \\
\dot{y}_1 &= 0
\end{aligned} .$$

Heteroclinic cycles

The join network supports the existence of four robust attracting simple heteroclinic cycles.

In the restriction to the synchrony subspace $P_1 \times \Delta_2$, the heteroclinic cycles:

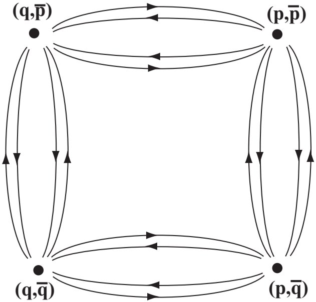
- $\mathcal{H}_1$, involving the two equilibria $(\mathbf{p}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{p}})$;
- $\mathcal{H}_2$, involving the two equilibria $(\mathbf{p}, \overline{\mathbf{q}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$.

In the restriction to the synchrony subspace $\Delta_1 \times P_2$, the heteroclinic cycles:

- $\mathcal{H}_3$, involving the two equilibria $(\mathbf{p}, \overline{\mathbf{p}})$ and $(\mathbf{p}, \overline{\mathbf{q}})$;
- $\mathcal{H}_4$, involving the two equilibria $(\mathbf{q}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$.

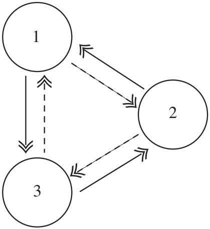
Heteroclinic network

The dynamics of the join network supports an attracting robust heteroclinic network $\mathcal{H} = \cup_{i=1}^{4} \mathcal{H}_i$ involving the partially synchronous equilibria $(\mathbf{p}, \overline{\mathbf{p}})$, $(\mathbf{p}, \overline{\mathbf{q}})$, $(\mathbf{q}, \overline{\mathbf{p}})$ and $(\mathbf{q}, \overline{\mathbf{q}})$,



contained in the union of the synchrony subspaces $\Delta_1 \times P_2$ and $P_1 \times \Delta_2$.

Guckenheimer and Holmes coupled cell systems

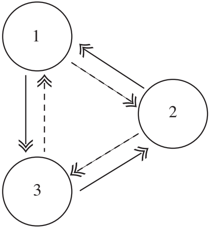


$$\begin{array}{l} \dot{x}_1 = f(x_1; x_2, x_3) \\ \dot{x}_2 = f(x_2; x_3, x_1) \\ \dot{x}_3 = f(x_3; x_1, x_2) \end{array} ,$$

with $f: \ \mathbf{R}^3 \to \mathbf{R}$ defined by

$$f(u; v, w) = u(1 + au^2 + bv^2 + cw^2), \quad a, b, c \in \mathbf{R}.$$

Guckenheimer and Holmes coupled cell systems



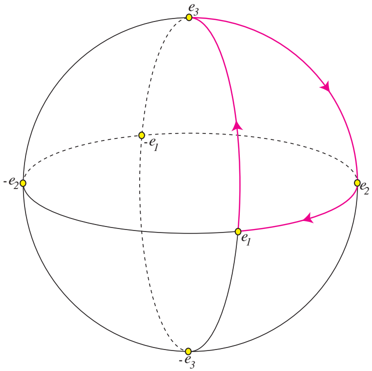
$$\begin{aligned}\dot{x}_1 &= f(x_1; x_2, x_3)\\ \dot{x}_2 &= f(x_2; x_3, x_1)\\ \dot{x}_3 &= f(x_3; x_1, x_2)\end{aligned} \quad ,$$

with $f: \mathbf{R}^3 \to \mathbf{R}$ defined by

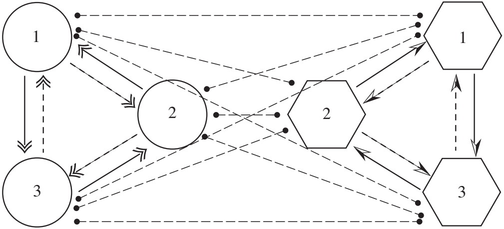
$$f(u; v, w) = u(1 + au^2 + bv^2 + cw^2), \quad a, b, c \in \mathbf{R}.$$

Flow invariant subspaces are all determined by the symmetry with $\{\mathbf{x} : x_1 = x_2 = x_3\}$ the only synchrony subspace.

Guckenheimer and Holmes heteroclinic cycle



Join of two Guckenheimer and Holmes systems



$$
\begin{aligned}
\dot{x}_1 &= x_1 \left(1 + a x_1^2 + b x_2^2 + c x_3^2 + d(y_1^2 + y_2^2 + y_3^2)\right)\\
\dot{x}_2 &= x_2 \left(1 + a x_2^2 + b x_3^2 + c x_1^2 + d(y_1^2 + y_2^2 + y_3^2)\right)\\
\dot{x}_3 &= x_3 \left(1 + a x_3^2 + b x_1^2 + c x_2^2 + d(y_1^2 + y_2^2 + y_3^2)\right)\\
\dot{y}_1 &= y_1 \left(1 + \overline{a} y_1^2 + \overline{b} y_2^2 + \overline{c} y_3^2 + d(x_1^2 + x_2^2 + x_3^2)\right)\\
\dot{y}_2 &= y_2 \left(1 + \overline{a} y_2^2 + \overline{b} y_3^2 + \overline{c} y_1^2 + d(x_1^2 + x_2^2 + x_3^2)\right)\\
\dot{y}_3 &= y_3 \left(1 + \overline{a} y_3^2 + \overline{b} y_1^2 + \overline{c} y_2^2 + d(x_1^2 + x_2^2 + x_3^2)\right)
\end{aligned} .
$$

Heteroclinic network in the join of two G&H systems

For $a, \overline{a}, b, \overline{b}, c, \overline{c}, d \in \mathbf{R}$, consider the following conditions:

$$b + c = 2a, \qquad \overline{b} + \overline{c} = 2\overline{a}, \qquad a\overline{a} - d^2 \neq 0,$$

$$\epsilon_a = \frac{d - a}{a\overline{a} - d^2} > 0, \qquad \epsilon_{\overline{a}} = \frac{d - \overline{a}}{a\overline{a} - d^2} > 0\,.$$

In the dynamics of the join coupled systems there is the four-dimensional flow invariant manifold

$$P = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^6 : \ \|\mathbf{x}\|^2 = \epsilon_{\overline{a}}, \quad \|\mathbf{y}\|^2 = \epsilon_a\right\}\ .$$

given by the product of two two-spheres with radius $\sqrt{\epsilon_{\overline{a}}}$ and $\sqrt{\epsilon_a}$, respectively.

Heteroclinic network in the join of two G&H systems

On the invariant manifold

$$P^+ = \{(\mathbf{x}, \mathbf{y}) \in P : \; x_i, y_i \geq 0, i = 1, 2, 3\}$$

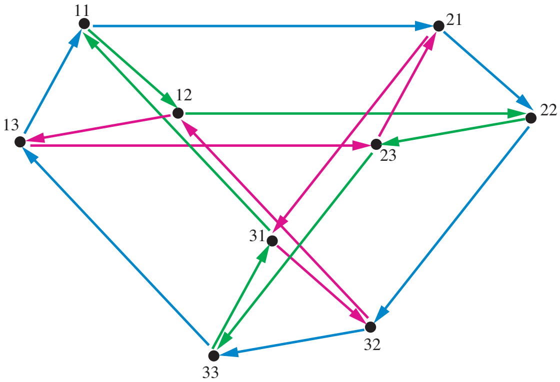
the dynamics of the join systems satisfy,

(a) There are nine saddle equilibrium points, e_{ij}, $i, j \in \{1, 2, 3\}$, with all coordinates equal to zero with the exception of the i-th and $(3 + j)$-th coordinates that are given by $\sqrt{\epsilon_{\overline{a}}}$ and $\sqrt{\epsilon_a}$, respectively.

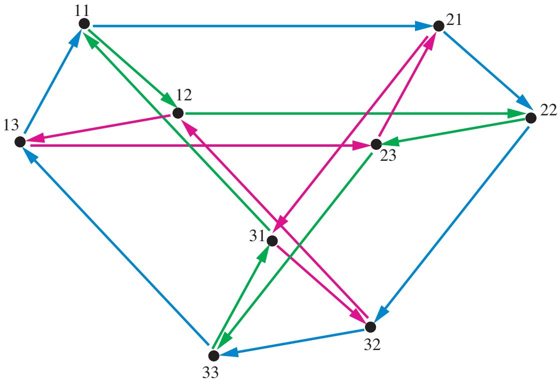
(b) There is a robust heteroclinic network involving the nine equilibria e_{ij}, $i, j \in \{1, 2, 3\}$ with one dimensional heteroclinic connections.

(c) Each equilibrium in the network is connected to other four equilibria, two by an incoming heteroclinic connection and the other two by an outcoming heteroclinic connection.

Heteroclinic network in the join of two G&H systems



Heteroclinic network in the join of two G&H systems



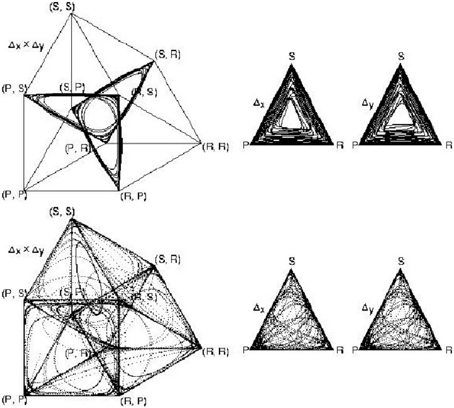
Same heteroclinic network structure as in the RSP-game!

Heteroclinic network in the join of two G&H systems

- The heteroclinic network structure is the same of the heteroclinic network in the RSP-game.
- Moreover, as in the case of the RSP-game, the sum of the contracting eigenvalues equals the sum of the expanding eigenvalues.

This implies switching in the neighbourhood of the network.

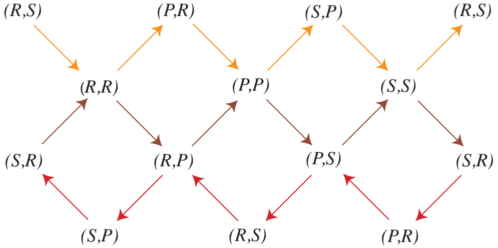
Numerical simulations - RSP game



[Y. Sato, E. Akiyama and J.P. Crutchfield] *Stability and diversity in collective adaptation*

Heteroclinic cycles in the RSP heteroclinic network

The network corresponds to the union of the following three heteroclinic cycles:



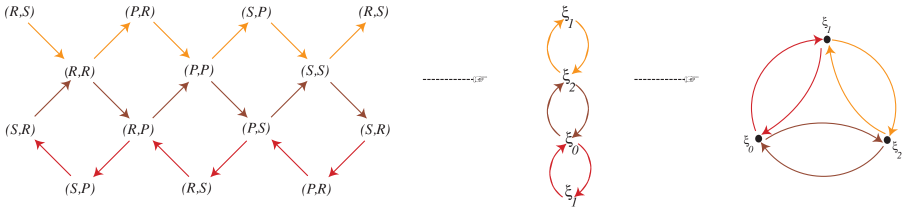
C_1 : $(R,S)\to(R,R)\to(P,R)\to(P,P)\to(S,P)\to(S,S)\to(R,S)$ (X wins or draw)

C_2 : $(S,R)\to(R,R)\to(R,P)\to(P,P)\to(P,S)\to(S,S)\to(S,R)$ (Y wins or draw)

C_0 : $(R,P)\to(S,P)\to(S,R)\to(P,R)\to(P,S)\to(R,S)\to(R,P)$ (X wins or Y wins)

A. and Castro, Chaotic switching in a two-person game, Physica D 239 (2010), 1598-1609.

Quotient heteroclinic network



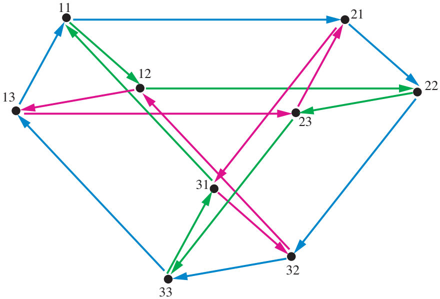
- We prove switching dynamics near the quotient network,
- lifting by the symmetry we get switching in the neighbourhood of the initial network.

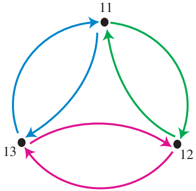
Switching in replicator dynamics and bimatrix games

- We consider edge heteroclinic networks for replicator dynamics and bimatrix games.
- We study the existence of switching near such networks.
- We prove that there is no switching in replicator dynamics.
- We give conditions for switching dynamics in bimatrix games.
- The mechanism for switching is conservative dynamics and no Kirk & Silber subnetwork.

A. Is there switching for replicator dynamics and bimatrix games? Physica D: Nonlinear Phenomena 240 (2011), 1475-1488.

Quotient of the heteroclinic network in the join of two G&H systems





Coupling of Guckenheimer and Holmes systems

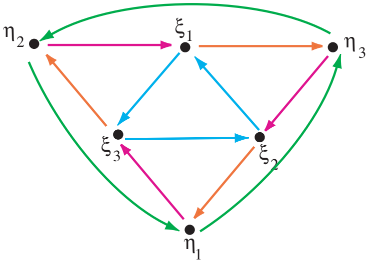
In [PD05] Postlethwaite and Dawes consider the coupling of two Guckenheimer & Holmes systems in a different way:

- two equal Guckenheimer & Holmes systems are coupled such that
- every cell in one system is connected to every cell in the other system and all those couplings are of different type.

$$
\begin{array}{l}
\dot{x}_1 = x_1 \left(\mu + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + b_1 y_1^2 + b_2 y_2^2 + b_3 y_3^2\right)\\
\dot{x}_2 = x_2 \left(\mu + a_1 x_2^2 + a_2 x_3^2 + a_3 x_1^2 + b_1 y_2^2 + b_2 y_3^2 + b_3 y_1^2\right)\\
\dot{x}_3 = x_3 \left(\mu + a_1 x_3^2 + a_2 x_1^2 + a_3 x_2^2 + b_1 y_3^2 + b_2 y_1^2 + b_3 y_2^2\right)\\
\dot{y}_1 = y_1 \left(\mu + a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2\right)\\
\dot{y}_2 = y_2 \left(\mu + a_1 y_2^2 + a_2 y_3^2 + a_3 y_1^2 + c_1 x_2^2 + c_2 x_3^2 + c_3 x_1^2\right)\\
\dot{y}_3 = y_3 \left(\mu + a_1 y_3^2 + a_2 y_1^2 + a_3 y_2^2 + c_1 x_3^2 + c_2 x_1^2 + c_3 x_2^2\right)
\end{array}
.
$$

[PD05] Postlethwaite and Dawes, *Regular and irregular cycling near a heteroclinic network*, Nonlinearity, 18 (2005), 1477-1509.

Heteroclinic network



- In [PD05] , for some parameter values it is observed what is calles regular cycling and irregular cycling.

- In both cases, a trajectory follows three cycles in the network in a sequential way.

- But, in the case of regular cycling, there is the same number n of turns around each cycle, whereas, in the irregular cycling, that number of turns varies in an irregular way.

- We note that each two cycles in the network that are visited sequentially have a heteroclinic connection in common.

[PD05] Postlethwaite and Dawes, *Regular and irregular cycling near a heteroclinic network*, Nonlinearity, 18 (2005), 1477-1509.

- The difference is that in the heteroclinic network in the RSP-game there are nine equilibria, and so, no Kirk and Silber subnetwork.
- The key ingredient for (infinite) switching in bimatrix games seems to be the existence of an edge heteroclinic network without a Kirk and Silber subnetwork in an incompressible flow.

Join of two equal Guckenheimer and Holmes systems

The system is equivariant by the symmetry group $\Gamma \ltimes \mathbf{Z}_2$, where the action of Γ is generated by k_x, k_y, ρ_x, ρ_y and the ction of $_2$ by

$$k_{xy}(x_1, x_2, x_3, y_1, y_2, y_3) = (y_1, y_2, y_3, x_1, x_2, x_3).$$

We have thus four more synchrony subspace (fixed-point subspace)

$$\begin{array}{l}
\{(\mathbf{x},\mathbf{y}) \in \mathbf{R}^6 : \ x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3\}, \\
\{(\mathbf{x},\mathbf{y}) \in \mathbf{R}^6 : \ x_1 = y_2,\ x_2 = y_3,\ x_3 = y_1\}, \\
\{(\mathbf{x},\mathbf{y}) \in \mathbf{R}^6 : \ x_1 = y_3,\ x_2 = y_1,\ x_3 = y_2\}, \\
\{(\mathbf{x},\mathbf{x}) \in \mathbf{R}^6\}.
\end{array}$$

Join of two equal Guckenheimer and Holmes systems

For $a, b, c, d \in \mathbf{R}$, consider the conditions

$$b + c = 2a, \qquad a + d < 0, \qquad d - a > 0,$$

which also imply that

$$\epsilon_a = -\frac{1}{a+d} > 0\,.$$

In the dynamics of the join coupled systems, under the conditions above, there is a four-dimensional flow invariant manifold given by the product of two 2-spheres with radius $\sqrt{\epsilon_a}$ that is globally attracting. The manifold

$$P = \left\{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^6 : \ \|\mathbf{x}\|^2 = \epsilon_a, \quad \|\mathbf{y}\|^2 = \epsilon_a\right\}. \qquad (1)$$

Join of two equal Guckenheimer and Holmes systems

On the invariant manifold P^+, the dynamics of the join systems satisfy,

(a) There are nine saddle equilibrium points, e_{ij}, $i, j \in \{1, 2, 3\}$, with all coordinates equal to zero with the exception of the i-th and $(3 + j)$-th coordinates that are given by $\sqrt{\epsilon_a}$.

(b) There is a robust heteroclinic network involving the nine equilibria e_{ij}, $i, j \in \{1, 2, 3\}$.

(c) Each equilibrium in the network is connected to other six equilibria, three by an incoming heteroclinic connection and three by an outcoming heteroclinic connection. Two of the incoming (outcoming) connections are one-dimensional and the other is two-dimensional.

THANKS!