On injectivity of locales and spaces

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joint work with Margarida Carvalho

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[D. Scott, LN, 1972]:

In Top_0, continuous lattices = spaces injective wrt embeddings

[P. Johnstone, JPAA, 1981]:

In Loc,
stably locally compact locales = retracts of coherent locales
= locales injective wrt flat embeddings

[M. Escardó, TA, 1998] and other papers:

In many examples of injectivity,

injective objects = Kan-injective objects = E.-M. algebras of a KZ-monad

Adding Kan-injectivity for morphisms, we obtain Kan-injective subcategories:

[Carvalho, S. TA, 2011]:

Kan-injective subcategories and KZ-monadic subcategories

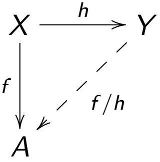
[Admek, S., Velebil, MSCS, 2015]:

An answer to the Kan-injective Subcategory Problem

Kan-injectivity

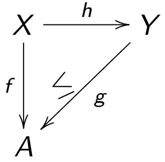
$\mathcal{X}$ order-enriched category

A is Kan injective wrt $h{:}X \to Y$ if A is injective wrt h and every $f : X \to A$ admits a left Kan extension f/h along h:



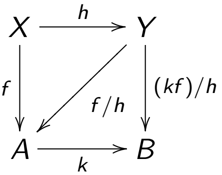
That is:

(1) $f = (f/h) \cdot h$

(2) If 

then $f/h \leq g$.

$A \xrightarrow{k} B$ is Kan-injective w.r.t. $X \xrightarrow{h} Y$ if A and B are so, and, for every $X \xrightarrow{f} A$, we have $(kf)/h = k(f/h)$:



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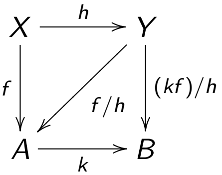

Given $\mathcal{H} \subseteq \mathrm{Mor}(\mathcal{X})$,

$\underbrace{\mathrm{KInj}(\mathcal{H})}_{\text{Kan-injective subcategory}}$:= subcategory of all objects and morphisms Kan-injective wrt $\mathcal{H}$

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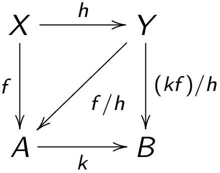

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Given $\mathcal{A}$ subcategory of $\mathcal{X}$,

$\mathcal{A}^{\mathrm{KInj}}$:= class of morphism wrt which $\mathcal{A}$ is Kan-injective

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Theorem

1) $\mathcal{A}$ is a KZ-monadic subcategory of $\mathcal{X}$, iff it is reflective

$$\mathcal{A} \underset{R}{\overset{I}{\rightleftarrows}} \mathcal{X} \qquad \textit{adjunction (order-enriched)}$$

with $R\eta \leq \eta R$, and $\mathcal{A}$ is closed under left adjoint retractions

$$\text{(i.e., if } \begin{array}{ccc} A & \xrightarrow{f\in\mathcal{A}} & B \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ X & \xrightarrow[g]{} & Y \end{array} \text{ with } e \text{ and } e' \text{ l. a. r. then } g \in \mathcal{A}\text{).}$$

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2) If $\mathcal{A}$ is a KZ-monadic subcategory of $\mathcal{X}$, then:

- $\mathcal{A}^{\mathrm{KInj}} = \{f \in \mathcal{X} \mid Rf \text{ left adj. section in } \mathcal{A}\} = \{f \in \mathcal{X} \mid Rf \text{ l. a. s. in } \mathcal{X}\}$
  $= R\text{-embeddings}$
- $\mathcal{A} = \mathrm{KInj}\,\{\eta_X \mid X \in \mathcal{X}\} = \mathrm{KInj}(\mathcal{A}^{\mathrm{KInj}})$

Examples

$\mathcal{X}$	Objs. of a KZ-monadic subcategory $\mathcal{A}$	$\mathcal{A}^{\mathsf{KInj}}$
Top_0	continuous lattices	embeddings
Top_0	continuous Scott domains	dense embeddings
Loc	stably locally compact locales	flat embeddings

In Loc

Loc = $\mathsf{Frm}^{\mathsf{op}}$

Locale = complete lattice L with $(\bigvee A) \wedge b = \bigvee_{a \in A} (a \wedge b)$, $A \subseteq L$, $b \in L$

Localic map = infima-preserving map $f : L \to M$ with $f^* : M \to L$ preserving finite meets

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Definition

$f : L \to M$ is n-flat, if $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$, for $|I| \leq n$.

(0-flat $=$) 1-flat $=$ dense ($f(0) = 0$)

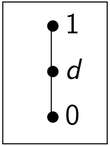
2-flat $=$ flat

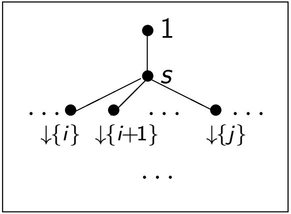
Free frame generated by a set X:

$SX = (\{\text{finite subsets of } X\}, \supseteq)$

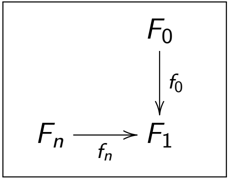
$$FX = \underbrace{(\{\text{downsets of } SX\}, \subseteq)}_{\text{free frame generated by } X}$$

For every cardinal n, F_n = frame generated by the set n

$F_1 =$ 

$F_n =$ 

$\downarrow\{i\} = \{A \subseteq n \mid A \text{ fin.}, i \in A\}$

$\mathcal{D}_n =$ 

with $f_n(1) = 1$, $f_n(s) = d$, and $f_n(x) = 0$ otherwise

Theorem

- *Embeddings* $= F_1^{\mathsf{KInj}}$
- *n-flat embeddings* $= \mathcal{D}_n^{\mathsf{KInj}}$

$\mathcal{D} = \bigcup_{n \in \mathsf{Card}} \mathcal{D}_n$ is a subcategory of Loc made of spatial locales.

Corollary

Loc *is the Kan-injective hull of* $\mathcal{D}$*, that is,*

$$\mathsf{Loc} = \mathsf{KInj}\left(\mathcal{D}^{\mathsf{KInj}}\right)$$

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Proof.

$$\mathcal{D}^{\mathsf{KInj}} = \bigcap_{n \in \mathsf{Card}} \mathcal{D}_n^{\mathsf{KInj}}$$

$$= \underbrace{\{f \in \mathsf{Loc} \,|\, f^* \in \mathsf{Loc} \text{ and } f^* f = \mathrm{id}\}}_{\mathcal{H}}$$

and $\mathsf{KInj}(\mathcal{H}) = \mathsf{Loc}$.

$L \in \mathrm{Loc}$

$G_2 L = \{U \subseteq L \mid U = \downarrow U$ and U closed under finite suprema$\}$

$a \ll b := \ \forall U \in G_2 L, \ b \leq \bigvee U \ \Rightarrow \ a \in U$

L is stably locally compact if

- $\forall a \in L, \ a = \bigvee_{x \ll a} x$
- $\forall x, \ a, \ b, \ (x \ll a, \ x \ll b) \Rightarrow x \ll a \wedge b$
- $1_L \ll 1_L$

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$$G_n : \mathrm{Loc} \to \mathrm{Loc}$$

gives rise to the functor part of a KZ-monad.

$\mathsf{SLComp_n} =$ category of stably locally n-compact locales and localic maps f such that f^* preserves $\ll_n$

Theorem

For every n, $\mathsf{SLComp_n}$ is a KZ-monadic subcategory, and it is the Kan-injective hull of $\mathcal{D}_n$, i.e.,

$$\mathsf{SLComp_n} = \mathsf{KInj}\left(\mathcal{D}_n^{\mathsf{KInj}}\right)^{\mathsf{KInj}}.$$

(Part of the proof makes an important use of [B. Banaschewski, Cahiers, 2005])

$$\mathsf{SLComp}_1 \underset{\neq}{\subset} \mathsf{SLComp}_2 \underset{\neq}{\subset} \mathsf{SLComp}_{\aleph_0} \subseteq \ldots$$

For $m < n$ with n regular, $\mathsf{SLComp}_\mathsf{m} \underset{\neq}{\subset} \mathsf{SLComp}_\mathsf{n}$

The union is Loc.

In Top_0

Order-enriched: for $X \underset{f}{\overset{g}{\rightrightarrows}} Y$, $f \leq g$ iff $f^{-1}(G) \supseteq g^{-1}(G)$, all open G

$$\begin{array}{ccc} \mathsf{Top}_0 & \xrightarrow{\mathsf{Lc}} & \mathsf{Loc} \\ X & \longmapsto & \Omega(X) \\ f & \longmapsto & (f^{-1})_* \end{array}$$

$\mathsf{Lc} \dashv \mathsf{Sp} : \mathsf{Loc} \to \mathsf{Top}_0$ order-enriched adjunction

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$\mathsf{Lc} \dashv \mathsf{Sp} : \mathbf{Loc} \to \mathbf{Top}_0$ order-enriched adjunction

$X \xrightarrow{f} Y$ is an embedding in $\mathbf{Top}_0$ iff $\mathsf{Lc}(f)$ is an embeding in $\mathbf{Loc}$

is dense in $\mathbf{Top}_0$ iff $\mathsf{Lc}(f)$ is dense in $\mathbf{Loc}$

is n-flat in $\mathbf{Top}_0$ iff $\mathsf{Lc}(f)$ is n-flat in $\mathbf{Loc}$

Proposition

Let $F \dashv G : \mathcal{A} \to \mathcal{X}$ be an order-enriched adjunction.
Then, given h in $\mathcal{X}$ and an object A (or a morphism f) in $\mathcal{A}$,

A (or f) is Kan-injective wrt Fh $\iff$ GA (or Gf) is Kan-injective wrt h.

Corollary

In Top$_0$*:*

- *Embeddings are precisely the morphisms wrt which the Sierpiński space S is Kan-injective, and*

  $$\mathsf{KInj}(S^{\mathsf{KInj}}) = \text{continuous lattices and maps preserving directed suprema and arbitrary infima.}$$

- *n-flat embeddings are precisely the morphisms wrt which $Sp[\mathcal{D}_n]$ is Kan-injective.*

Denoting the Kan-injective hull by $\mathbb{K}$, i.e., $\mathbb{K}(-) = \mathsf{KInj}(-^{\mathsf{KInj}})$, we have:

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$\mathbb{K}(\mathsf{Sp}[\mathcal{D}_2]) =$ stably locally compact spaces and continuous maps f such that $U \ll V \;\Rightarrow f^{-1}(U) \ll f^{-1}(V)$.

$\mathbb{K}(\mathsf{Sp}[\mathcal{D}_1]) =$ continuous Scott domains and maps preserving directed suprema and arbitrary non-empty infima.

$\mathbb{K}(S) =$ continuous Scott domains and maps preserving directed suprema and arbitrary infima.