Wind Finslerian structures: from Zermelo's navigation to the causality of spacetimes

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(joint work with E. Caponio and M.A. Javaloyes)

Finsler Geometry

Classical Finsler Geometry:

- (M, g_R) Riemannian: replace Euclidean scalar products by (positively homogeneous) norms at each $p \in M$
  Positively homogeneous: $\| \lambda v \| = |\lambda| \| v \|$ for $\lambda \geq 0$

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- Motivation: norms more general than Euclidean scalar products (parallelogram identity)
- Application: Lagrangians, navigation, fastest trajectories to go up and down a hill...

Two types of Lorentz-Finsler links

Link 1 with Lorentzian Geometry:

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We will NOT follow this link

Two types of Lorentz-Finsler links

Link 2: pure geometric correpondence between

- A class of spacetimes $\longleftrightarrow$ A class of Finsler manifols

Understanding the link 2

1. Product spacetime $(\mathbb{R} \times M, g_L = -dt^2 + g_0)$, g_0 Riemannian:
   Finsler $F(v) = \sqrt{g_0(v, v)}$, $v \in TM$.
   Lightlike directions $\longleftrightarrow$ *F-unit vectors*

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2. Standard static spacetimes $(\mathbb{R} \times M, -\Lambda dt^2 + g_0)$, $\Lambda > 0$ (on M): Finsler $F(v) = \sqrt{g_0(v,v)/\Lambda}$.
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3. Product spacetime with stationary cross term $(\mathbb{R} \times M, g_L = -dt^2 + \omega \otimes dt + dt \otimes \omega + g_0)$, ω: 1 form.
   Finsler (Randers): $F^{\pm}(v) = \sqrt{g_0(v,v) + \omega(v)^2} \pm \omega(v)$.
   Future (resp. past)-directed lightlike directions $\longleftrightarrow$ *F^+ (resp. F^-)-unit vectors*

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4. Standard stationary (strictly)
   $(\mathbb{R} \times M, g_L = -\Lambda dt^2 + \omega \otimes dt + dt \otimes \omega + g_0)$, $\Lambda > 0$.
   Finsler (Randers): $F^{\pm}$ for g_L/Λ.
   Future (resp. past)-directed lightlike directions
   $\longleftrightarrow$ *F^+ (resp. F^-)-unit vectors*

Understanding the link 2

Step further to be developed here

- Standard space-transverse Killing (SSTK):
  $(\mathbb{R} \times M, g_L = -\Lambda dt^2 + \omega \otimes dt + dt \otimes \omega + g_0)$,
  only under $\Lambda + \| \omega \|^2 > 0$ (Lorentzian restriction).
  Assign "wind-Finsler structures" $\Sigma^{\pm}$ so that
  Future (resp.) past-directed lightlike directions
  $\longleftrightarrow \Sigma^+$ *(resp.* Σ^-*)-unit vectors*

Stationary to Randers

Previous results on the standard stationary case
(Stationary spacetimes vs Randers spaces)

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   (Caponio, Germinario, —, J. Geom. Anal.'15)
3. Causal boundaries $\longleftrightarrow$ Cauchy, Gromov and Busemann boundaries in Finslerian (and Riemannian) settings
   (Flores, Herrera, — Memoirs AMS'13).

And so on...

Summing up: precedents

Conformal structure of a class of spacetimes: (standard) **stationary** *ones*
$\longleftrightarrow$ *Geometry of a class of Finsler manifolds:* **Randers** *spaces*

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—some of them extensible to general Finsler manifolds
- $\leftarrow$ Finsler elements allow a precise description of spacetime counterparts

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Broader relation

Lorentzian Geometry $\longleftrightarrow$ *Finsler Geometry*

(including the Riemannian one!)

Summing up: aim

A step further: equivalence between

Conformal structure of a class of spacetimes:
(standard) **space-transverse Killing** *(SSTK) ones* $\longleftrightarrow$
Geometry of a class of **generalized Finsler** *manifolds:*
Wind Riemannian/ Wind Finslerian *structures*

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Wind Riemannian/ Wind Finslerian *structures*

Applicability:

- $\rightarrow$ new geometric elements and results for wind Riemannian structures can be obtained from the spacetime viewpoint —some of them extensible to general wind Finsler structures
- $\leftarrow$ (generalized) Finsler elements allow a precise description of spacetime counterparts.

Summing up: aim

Remarkably:

- Wind Riemannian structures include some "singular Finsler geometries" commonly used (Kropina metrics), which are described by "non-singular" spacetimes.

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- Wind Riemannian structures include some "singular Finsler geometries" commonly used (Kropina metrics), which are described by "non-singular" spacetimes.
- Broader relation:
  Lorentzian Geometry $\longleftrightarrow$ Extended Finsler Geometry

Non-relativistic motivation

Why to generalize Finsler manifolds?

Motivation: navigation and spacetimes

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- *Note:* in the literature "Finsler" is commonly used for non-standard notions of Finsler manifolds

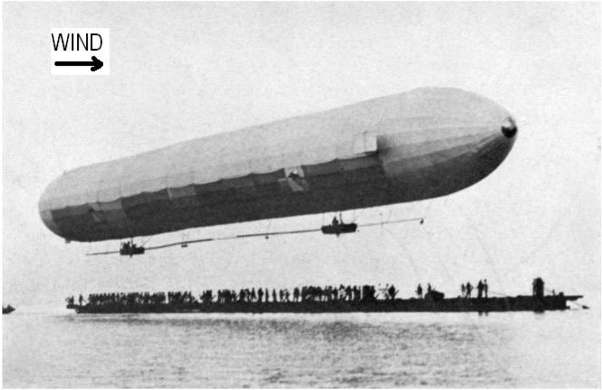
Non-relativistic motivation

Why to generalize Finsler manifolds?

- *Note:* in the literature "Finsler" is commonly used for non-standard notions of Finsler manifolds
- A simple example on the necessity of our generalization: *windy navigation*

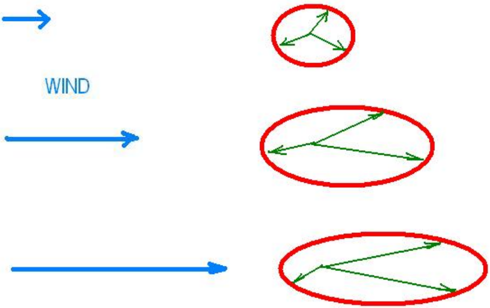
Motivation: navigation and spacetimes

Classical Zermelo's navigation: plane/Zeppelin in the air or ship on the sea with a (mild) wind.



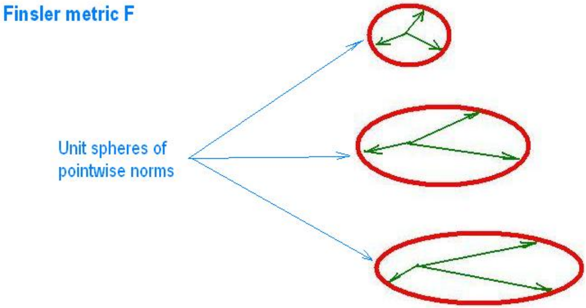
Motivation: navigation and spacetimes

The possible maximum velocities at each point and direction (linearized trips of unit time) determine a (topological) smooth sphere Σ_p at each tangent space $T_pM, p \in M$



Motivation: navigation and spacetimes

Regarding the spheres Σ_p as the indicatrices (unit spheres) for (non-reversible) norms, a Finsler metric Z (Zermelo) is obtained



Note:

- The geodesics for Z are the curves that (locally) minimize the time of the trip between pairs of points

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  1. taking a Riemannian metric g_R and
  2. shifting the centers of the unit balls by means of a vector field W (wind)

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- Zermelo metric is, in fact, a Randers metric obtained by:
  1. taking a Riemannian metric g_R and
  2. shifting the centers of the unit balls by means of a vector field W (wind) ... with $g_R(W, W) < 1$

Motivation: navigation and spacetimes

What about if the wind is not mild? $(g_R(W, W) \geq 1)$

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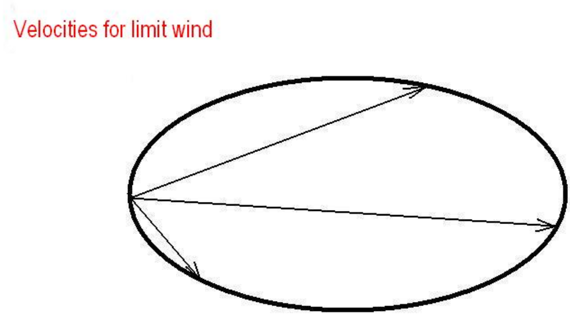
- The plane/ship is not able to move in some forbidden directions:
  some regions become unreachable —or must be reached by going around

Motivation: navigation and spacetimes

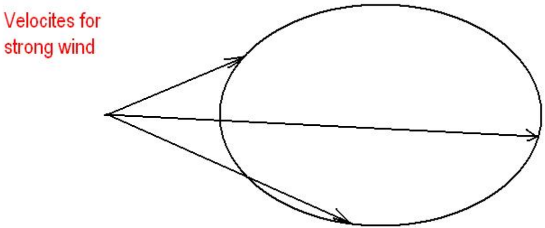
In the critical case $g_R(W, W) = 1$ one obtains a Kropina metric, which is singular as a Finsler metric



Motivation: navigation and spacetimes

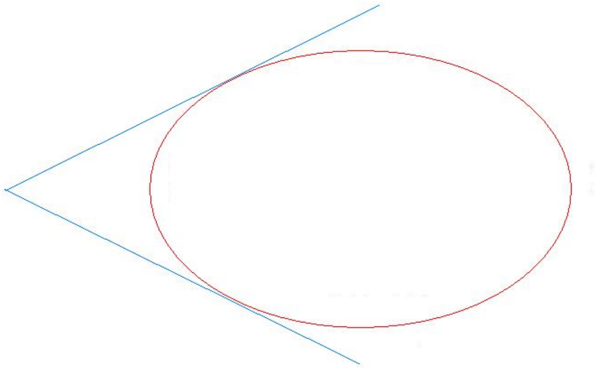
For strong wind $g_R(W, W) > 1$,

- Vector 0 does not belong to the "unit ball"



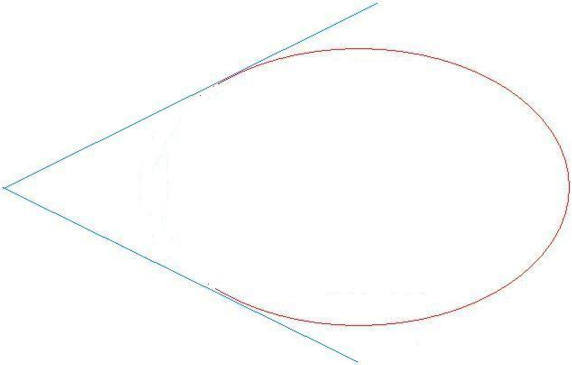
Motivation: navigation and spacetimes

From the Finsler viewpoint, one has two "conic Finsler pseudometric":



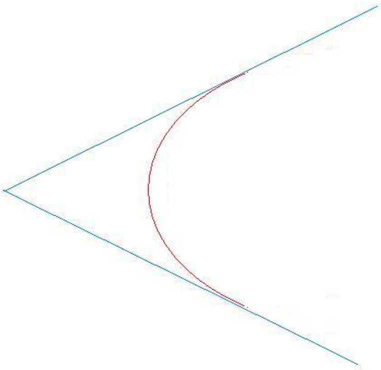
Motivation: navigation and spacetimes

- one properly Finslerian ("definite positive", convex indicatrix)



Motivation: navigation and spacetimes

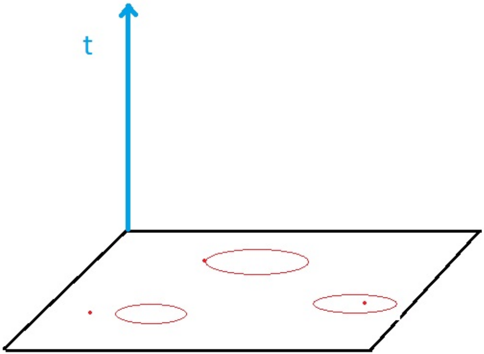
- the other Lorentzian (concave).



This seems complicated! ...but, from the viewpoint of the set Σ of the indicatrices, nothing singular happens.

The spacetime viewpoint:

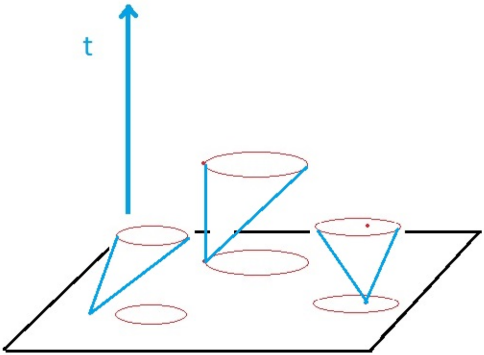
- Add the time as a dimension more



Motivation: navigation and spacetimes

The spacetime viewpoint:

- Add the time as a dimension more
- Putting a “unit of time” to all the indicatrices... one has a cone structure



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   The metric does not change with t (∂_t is Killing) (Notice that the wind is not assumed to vary with time, i.e., the situation is "stationary")
2. One can visualize the reachable regions... as well as those regions that must be abandoned necessarily.
   This is completely analogous to the situations for causal futures, black holes and all the relativists' fauna.

Motivation: navigation and spacetimes

The moral is then:

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The moral is then:

1. One has powerful tools to describe Zermelo's navigation by using spacetimes, in a smooth non-singular way, including Kropina metrics!
2. But this will be useful to describe spacetimes too: the "conformal initial data" ($t = 0$) that determines the Lorentzian metric are the introduced "wind Finsler" elements. That is, the conformal part of the so-called *Killing initial data* for Einstein equations can be always represented by "wind Finsler" elements!

Parts of the remainder of the talk

1. Wind Finslerian structures
2. SSTK spacetimes
3. Mild and critical wind: Randers-Kropina (Causal K)
4. Arbitrary wind and wind Riemannian structures

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Finsler metrics

Definition

Finsler metric $F : TM \to \mathbb{R}$:
— positively homog. strongly convex norm at each $p \in M$
— varying with p continuously and smooth away 0.

- **Strongly convex**: the second fundamental form of the unit sphere is positive definite
  —if replaced by concaveness, Lorentz-Finsler
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  —if replaced by concaveness, Lorentz-Finsler
  (but necessarily defined only in a *conic* domain of TM)
- Positively homogeneous: $F(\lambda v) = \lambda F(v)$ for $\lambda > 0$
- Reversed Finsler metric: $F^{\text{rev}}(v) := F(-v)$

Distance and balls:

Taking infimum of lengths of curves connecting two points, each Finsler metric induces a *generalized distance d_F*. This means:

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Centered at any point x_0, there are:

- forward balls: $d_F(x_0, x) < r$
- backward balls: $d_F(x, x_0) < r$

They may differ but each one generate the manifold topology.

Finsler metrics

Relevant examples, for Riemannian g_R, 1-forms ω, β:

- Randers metric: $R = \sqrt{g_R + \omega^2} + \omega$ $(= \sqrt{h} + \omega, \| \omega \|_h < 1)$

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Relevant examples, for Riemannian g_R, 1-forms ω, β:

- Randers metric: $R = \sqrt{g_R + \omega^2} + \omega$ $(= \sqrt{h} + \omega, \| \omega \|_h < 1)$
- Kropina metric: $F = g_R / \beta$
  —Defined in the *conic* domain (open half plane)
  $\beta(v) > 0$
  —It will be a "limit case" of Randers
  – Beware! One can define formally d_F but
  typically $d_F(x, x) = \infty$.

Notion of wind Finslerian structure

Definition

For a vector space V:

—**Wind Minkowski structure:** Compact strongly convex smooth hypersurface Σ^V embedded in V

—Unit ball B Bounded open domain B enclosed by Σ^V

—Conic domain A : open region determined by B from 0.

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For a manifold M:
— **Wind Finsler structure:** smooth hypersurface $\Sigma \hookrightarrow TM$: $\Sigma_p = \Sigma \cap T_pM$ is wind Minkowski in T_pM (+transversality)
— Ball at p: $B_p \subset T_pM$ ($\rightsquigarrow A_p$) Domain $A := \cup_p A_p$
— Region of strong wind: $M_l := \{p \in M : 0 \notin \bar{B}_p\}$
— Properly conic domain: $A_l := \Sigma \cap \pi^{-1}(M_l)$

Notion of wind Finslerian structure

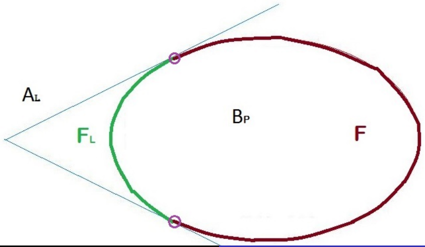
Proposition

Any Σ determines two conic pseudo-Finsler metrics:

(i) $F : A \to [0, +\infty)$ *conic Finsler metric on all* M,

(ii) $F_l : A_l \to [0, +\infty)$ F_l *is a Lorentz-Finsler metric in the region* M_l *of strong wind*

Moreover, $F < F_l$ *on* A_l, F *and* F_l *can be extended continuously, and both extensions coincide on the boundary of* A_l



Notion of wind Finslerian structure

Proposition

Any Σ is the displacement of the indicatrix of Finsler metric F_0 along some vector field W:

$$F_0\left(\frac{v}{Z(v)} - W\right) = 1,$$

($v \in \Sigma \iff Z(v)$ is a solution)

Wind Riemann structures

Definition

A wind Riemann structure is a wind Finslerian structure $\Sigma \subset TM$ such that, alternatively:

- Σ is the translation of the indicatrix of a Riemannian norm $F_0 = \sqrt{g_R}$ along some vector field W.
- Σ_p is an ellipsoid $\forall p \in M$.

Wind Riemannian structures

Proposition

Let (M, Σ) be a wind Riemann structure. Then, for some h semi-Riemannian and β 1-form

(i) $0_p \in B_p \Rightarrow \Sigma_p$ *indicatrix for Randers:*
$F(v) = \sqrt{h(v,v)} + \beta(v)$
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(ii) $0_p \in \Sigma_p \Rightarrow \Sigma_p$ *indicatrix for Kropina:*
$F(v) = -h(v,v)/2\beta(v)$ *(on $A_p = \{v \in T_pM : -\beta(v) > 0\}$),*

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(i) $0_p \in B_p \Rightarrow \Sigma_p$ *indicatrix for Randers:*
$F(v) = \sqrt{h(v,v)} + \beta(v)$
(h *Riemann on* $A_p = T_pM$, $\|\beta\|_h < 1$*),*

(ii) $0_p \in \Sigma_p \Rightarrow \Sigma_p$ *indicatrix for Kropina:*
$F(v) = -h(v,v)/2\beta(v)$ *(on* $A_p = \{v \in T_pM : -\beta(v) > 0\}$*),*

(iii) $0_p \notin \bar{B}_p \Rightarrow \Sigma_p$ *indicatrix for two pseudo-Randers type:*
$F(v) = -\sqrt{h(v,v)} - \beta(v), F_l(v) = \sqrt{h(v,v)} - \beta(v)$
$A_p = \{v \in T_pM : h(v,v) > 0$ *and* $-\beta(v) > 0\}$.
$-h$ *Lorentz,* $\beta(v)^2 > \alpha(v,v)$, $v \in T_pM \backslash 0$.

Wind Riemannian structures

Remark.

- The elements h, β cannot be chosen to match continuously in the Kropina region $\rightsquigarrow$ complicated viewpoint
- However, this is possible when $0 \in \bar{B}_p, \forall p \in M$ $\rightsquigarrow$ "Randers-Kropina" metric appears:

  lengths, distances, geodesics *formally* definible as a limit of the standard Finslerian case...
  (but with striking differences!)

Parts of the talk

1. Wind Finslerian structures
2. **SSTK spacetimes**
3. Mild and critical Wind: Randers-Kropina (Causal K)
4. Arbitrary wind and wind Riemannian structures

Notion of SSTK

A spacetime $(\mathbb{R} \times M, g)$ is *standard with a space-transverse Killing vector field (SSTK)* when

$$g = -(\Lambda \circ \pi)\mathrm{d}t^2 + \pi^*\omega \otimes \mathrm{d}t + \mathrm{d}t \otimes \pi^*\omega + \pi^* g_0,$$

(necessarily $\Lambda > -\|\omega\|_0^2$) $\pi : \mathbb{R} \times M \to M$ projection

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$$\partial_t \ \text{Killing and} \begin{cases} \text{timelike} & \Lambda > 0 \\ \text{lightlike} & \Lambda = 0 \\ \text{spacelike} & \Lambda < 0 \end{cases}$$

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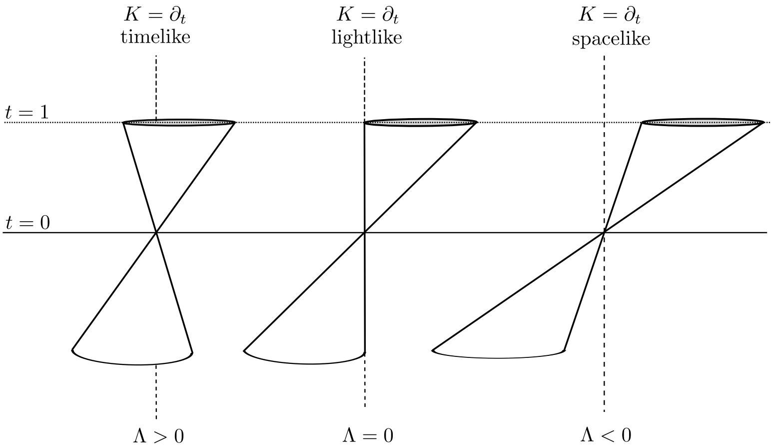
The projection $t : \mathbb{R} \times M \to \mathbb{R}$ temporal function
[for v causal (timelike or lightlike), $dt(v) > 0$ defines the future direction]

Notion of SSTK



Wind Riemannian structure for a SSTK

The set of all the future-pointing lightlike directions determines:
$\Sigma = \{v \in TM : (1, v) \text{ is (future-p.) lightlike in } T(\mathbb{R} \times M)\}$

Proposition

Σ is a wind Riemannian structure
*—the Fermat structure of the **conformal** class of the SSTK*

Proof. $(1, v)$ lightlike iff $-\Lambda + 2\omega(v) + g_0(v, v) = 0$,
—pointwise ellipsoid by the Lorentzian condition $\Lambda > -\|\omega\|_0^2$. ■

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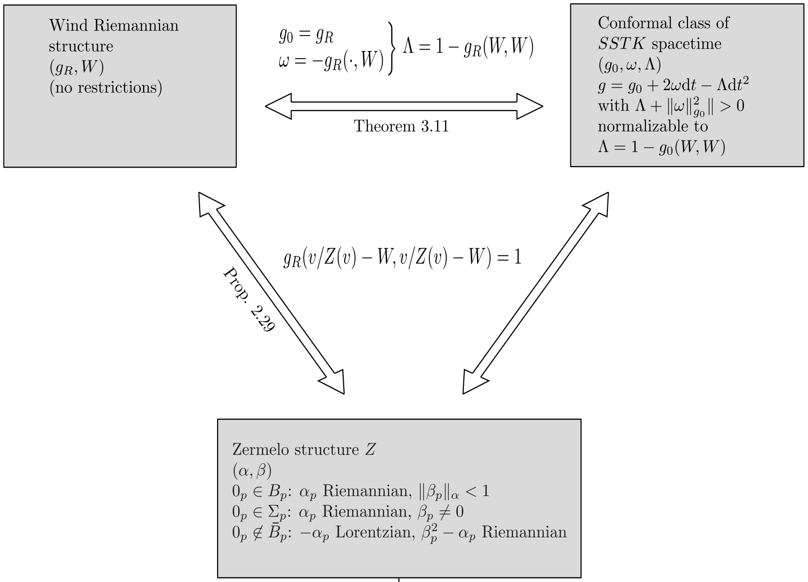
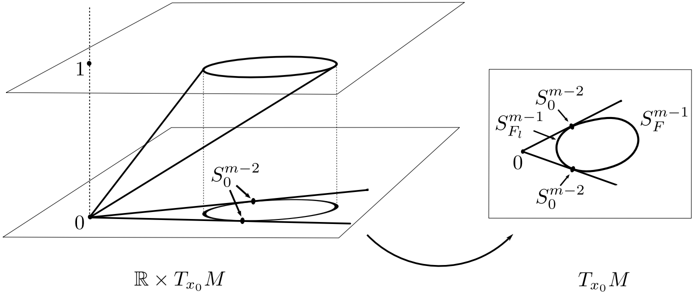
Remark: analogously for past: $\tilde{\Sigma} = -\Sigma$



Wind Riemannian structure
(g_R, W)
(no restrictions)
$g_0 = g_R$
$\omega = -g_R(\cdot, W)$
$\Lambda = 1 - g_R(W,W)$
Theorem 3.11
Conformal class of
$SSTK$ spacetime
(g_0, ω, Λ)
$g = g_0 + 2\omega dt - \Lambda dt^2$
with $\Lambda + \|\omega\|^2_{g_0}\| > 0$
normalizable to
$\Lambda = 1 - g_0(W,W)$
Prop. 2.29
$g_R(v/Z(v) - W, v/Z(v) - W) = 1$
Zermelo structure Z
(α, β)
$0_p \in B_p$: α_p Riemannian, $\|\beta_p\|_\alpha < 1$
$0_p \in \Sigma_p$: α_p Riemannian, $\beta_p \neq 0$
$0_p \notin \bar{B}_p$: $-\alpha_p$ Lorentzian, $\beta_p^2 - \alpha_p$ Riemannian

Interpretation of F and F_l

Region $M_l \equiv \{\Lambda < 0\}$ (∂_t spacelike)



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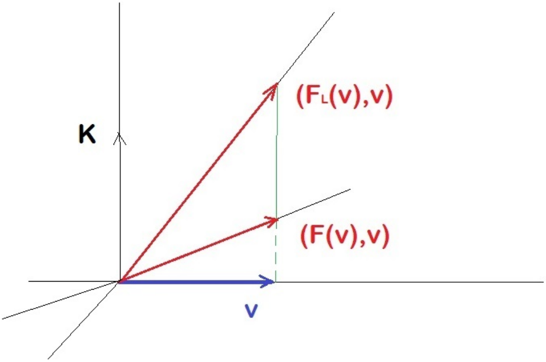

Interpretation of F and F_l

Explicit formulas:

$$F(v) = \frac{g_0(v,v)}{-\omega(v)+\sqrt{\Lambda g_0(v,v)+\omega(v)^2}}, \quad \forall v \in A$$
$$F_l(v) = -\frac{g_0(v,v)}{\omega(v)+\sqrt{\Lambda g_0(v,v)+\omega(v)^2}}, \quad \forall v \in A_l,$$

Interpretation of the metric in the root:

$$h(v,v) = \Lambda g_0(v,v) + \omega(v)^2$$

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- h well defined on all M

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- h well defined on all M

Proposition

Let $p_{\mathbb{R}}^{\perp} : T(\mathbb{R} \times M_{\Lambda \neq 0}) \to T(\mathbb{R} \times M_{\Lambda \neq 0})$ the natural projection on the bundle $\partial_t^{\perp}$ g-orthogonal to ∂_t.

$$h(v,v) = -\Lambda g(p_{\mathbb{R}}^{\perp}(0,v), p_{\mathbb{R}}^{\perp}(0,v)) \quad \forall v \in T_x M, x \in M_{\Lambda \neq 0}$$

Interpretation of F and F_l

Summing up:
$-h/\Lambda$ is the metric for the projection on $K^{\perp}$ $(= \partial_t^{\perp})$ up to Λ, which allows its extension to all M

$$h \begin{cases} \text{Riemannian} & \text{if } \Lambda > 0 \\ \text{Lorentz with index } n-1 & \text{if } \Lambda < 0 \\ \text{Degenerate} & \text{if } \Lambda = 0 \end{cases}$$

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Causal K

Case

$$K = \partial_t \quad \begin{cases} \text{Causal (timelike or lightlike)} & \Lambda \geq 0 \\ \text{Non-strong wind (mild or critical)} & g_R(W, W) \leq 1 \\ \text{Randers-Kropina metric} & F \end{cases}$$

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d_F : F-separation , formal distance but POSSIBLY:

- Non- symmetric
- $d_F(x, x) > 0$
- $d_F(x, y) = +\infty$ (even for $x = y$)

Description of chronology in terms of d_F-balls

Characterization of chronology

Description of chronology in terms of d_F-balls

Proposition

For any SSTK spacetime $(\mathbb{R} \times M, g)$ *with causal* K*:*

$$(t_0, x_0) \ll (t_1, x_1) \quad \Leftrightarrow \quad d_F(x_0, x_1) < t_1 - t_0$$

$$I^+(t_0, x_0) = \{(t, y) : d_F(x_0, y) < t - t_0\},$$
$$I^-(t_0, x_0) = \{(t, y) : d_F(y, x_0) < t_0 - t\}.$$

Continuity of d_F K

Properties for d_F from non-trivial properties of limits of lightlike geodesics

Continuity of d_F K

Properties for d_F from non-trivial properties of limits of lightlike geodesics

Proposition

For any Randers-Kropina metric F on M:

$d_F : M \times M \to [0, \infty]$ *is continuous away from the diagonal*

$D = \{(x, x) : x \in M\} \subset M \times M.$

Full characterization of causality

Main theorem:
For any SSTK $(\mathbb{R} \times M, g)$ (necessarily stably causal) with causal K

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1. (M, F) is *convex*: $x_0, x_1 \in M$ with $d_F(x_0, x_1) < \infty$ connectable by minimizing geodesic.
2. $J^{+}(p)$ *is closed* $\forall p \in \mathbb{R} \times M$.
3. $J^{-}(p)$ *is closed* $\forall p \in \mathbb{R} \times M$.

Full characterization of causality

- Global hyperbolicity ($J^+(p) \cap J^-(q)$ compact) equivalent to All $\bar{B}_F^+(x_0, r_1) \cap \bar{B}_F^-(x_1, r_2)$ compact

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  1. Closures $\bar{B}_F^+(x, r)$, $\bar{B}_F^-(x, r)$ compact
  2. F forward and backward geodesically complete. (i.e. all geodesics extendible to $+\infty$ and $-\infty$)

Further consequences for d_F

Straightforward consequence: Hopf-Rinow type theorem

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Corollary

For any Randers-Kropina metric F on a manifold M:

- *Forward geodesic completeness of F*
  $\Longleftrightarrow$ *compactness of $\bar{B}^+(x, r)$ (forward closed balls)*
- $\Longrightarrow$ *compactness of $\bar{B}^+(x_1, r_1) \cap \bar{B}^-(x_2, r_2)$*
- $\Longrightarrow$ *convexity of (M, F)*

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Balls and geodesics for wind Finsler

No “distance” d_F for wind Riemannian

$\rightsquigarrow$ *redefinitions of balls and geodesics for any wind Finsler –simplify here always by using “wind curves” (w.c.) with velocity in or inside the indicatrices*

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Wind balls:

$$B^+_\Sigma(x_0, r) = \{x \in M \mid \exists\, \gamma \text{ w. c. } : \ell_F(\gamma) < r < \ell_{F_l}(\gamma)\},$$
$$B^-_\Sigma(x_0, r) = \{x \in M \mid \exists\, \gamma \text{ w. c. } : \ell_F(\gamma) < r < \ell_{F_l}(\gamma)\}.$$

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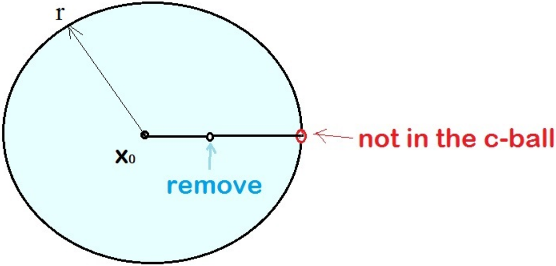
Wind c-balls:

$$\hat{B}^+_\Sigma(x_0, r) = \{x \in M \mid \exists\, \gamma \text{ w. c. } : \ell_F(\gamma) \leq r \leq \ell_{F_l}(\gamma)\},$$
$$\hat{B}^-_\Sigma(x_0, r) = \{x \in M \mid \exists\, \gamma \text{ w. c. } : \ell_F(\gamma) \leq r \leq \ell_{F_l}(\gamma)\}.$$

Balls and geodesics for wind Finsler

Closed balls: $\bar{B}^+_\Sigma(x_0, r)$, $\bar{B}^-_\Sigma(x_0, r)$

$$B^+_\Sigma(x_0, r) \subset \hat{B}^+_\Sigma(x_0, r) \subset \bar{B}^+_\Sigma(x_0, r)$$



Balls and geodesics for wind Finsler

Extremizing (wind) pregeodesic: given $x_1 \in \hat{B}^+_\Sigma(x_0, r) \setminus B^+_\Sigma(x_0, r)$, w. c. curve γ: $\ell_F(\gamma) \leq r \leq \ell_{F_l}(\gamma)$

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Pregeodesic: locally extremizing pregeodesic
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Proposition

Let γ geodesic parametrized by arc length for a wind Finslerian structure (M, Σ): $\dot{\gamma}(\mathbf{t}) \in \mathbf{A}$ (open)
Then γ is a (standard) unit geodesic for either F or F_l.

Characterization of causal relations in SSTK

Proposition

Let $(\mathbb{R} \times M, g)$ *a SSTK:*

$$I^+(t_0, x_0) = \cup_{s>0} \{t_0 + s\} \times B^+_\Sigma(x_0, s),$$

$$J^+(t_0, x_0) = \cup_{s\geq 0} \{t_0 + s\} \times \hat{B}^+_\Sigma(x_0, s)$$

Moreover:

$$(t_1, x_1) \in J^+(t_0, x_0) \setminus I^+(t_0, x_0) \iff$$

$$\exists\gamma \textit{ extremizing geodesic from } x_0 \textit{ to } x_1 : \begin{cases} \ell_F(\gamma) = t_1 - t_0 \\ \ell_{F_l}(\gamma) = t_1 - t_0 \end{cases}$$

Consequences for wind Riemannian

Characterization of wind geodesics, solution of navigation (extending Bao, Robles, Shen '04) and "normal" neighborhoods

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Theorem

Let (M, Σ) be a wind Riemannian structure. A wind curve x in M_l is a wind geodesic $\iff$ x is

- *a (standard) geodesic of F, F_l or*
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Moreover, given $x_0 \in M$, $\exists \epsilon > 0$:

- *the c-balls $\hat{B}^{\pm}(x, r)$ are compact, and*
- *the F (resp. F_l) geodesics parametrized by the arc length departing from x_0 are defined on all $[0, r]$ and are global minima (resp. local maxima).*

Main theorem: for any SSTK $(\mathbb{R} \times M, g)$

- Stable causality (existence of a temporal function) holds always

Full characterization of causality

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- Causal simplicity equivalent to: w-convexity : the c-balls are closed

Full characterization of causality

- Global hyperbolicity equivalent to any of
  1. All intersections $\bar{B}^+_\Sigma(x_0, r_1) \cap \bar{B}^-_\Sigma(x_1, r_2)$ compact
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  3. Σ forward and backward geodesically complete .

Further applications

1 Hopf-Rinow properties:
generalizing the Randers-Kropina case

Further applications

1. Hopf-Rinow properties:
   generalizing the Randers-Kropina case
2. Accessible regions for Σ **(including planes and ships)** and K-horizons:
   determined by $-h$ (in M_l and its "Newtonian limit" in $\Lambda = 0$)

Further applications

3 Cauchy developments:
fully characterization in terms of the “wind-distance” to a subset
$\rightsquigarrow$ properties of smoothness of the distance to a subset

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- Riemannian case: horizons in a product spacetime (Chrusciel, Fu, Galloway and Howard ’02)
- Finslerian case (connections with Hamilton-Jacobi, Li, Nirenberg ’05): horizons in a standard stationary spacetime (Caponio, Javaloyes, — ’11)
- Randers-Kropina and properly Wind Finsler: further results, including interpretations and generalizations of results for trapped surfaces (Mars and Reiris ’12).

Further applications

4 Fermat principle:
holds for the arrival time to integral curves of $K = \partial_t$ even when they are spacelike!

5 Existence of closed geodesics

- Muito obrigado pela vossa atenção
- Thank you very much for your attention
- Muchas gracias por vuestra atención