Defects, Orbifolds and Spin

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joint with S. Novak

Outline

Idea:

Do the state-sum construction of an n-dimensional topological field theory inside an n-dimensional quantum field theory with defects.

Here: Restrict to 2d theories (only dimension with worked-out examples)

Results:

- Orbifolds from defects
- Spin from defects

Plan:

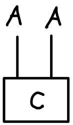
- "...state-sum construction ..."
- "...QFT with defects."
- "...inside ..."
- apply to 2d CFT

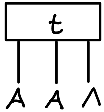
State-sum construction of 2d topological field theories

Bachas, Petropoulos '92
Fukuma, Hosono, Kawai '92

Data: fin. dim. k-vector space A , $c \in A \otimes A$, $t : A \otimes A \otimes A \to k$.

More general: take A, c, t in a symmetric monoidal category



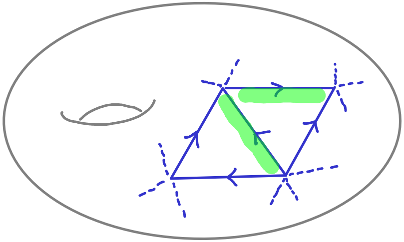


Σ : oriented surface for brevity: surfaces with empty boundary only

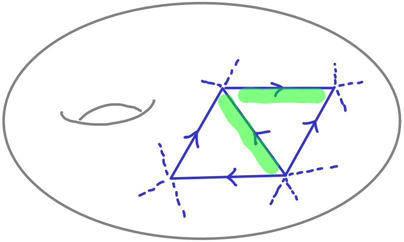
To evaluate 2d state sum TFT on Σ ,

1. Pick a triangulation of Σ . Equip with a *marking*: marked side for each triangle & orientation for each edge.

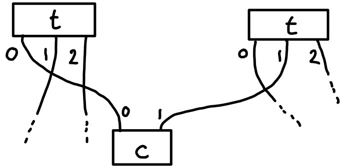
Σ =



. . . state-sum construction

$\Sigma =$ 

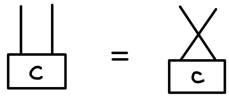
2. Use combinatorial data to build a linear map, e.g.

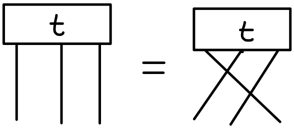


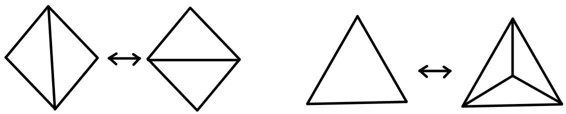
Get $Z(\Sigma) = t^{\otimes(\# triang)} \circ (permutation) \circ c^{\otimes(\# edges)}$

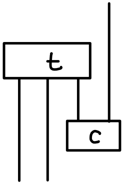
... state-sum construction

Independence of choice of marked triangulation from independence under *local moves*:

edge or. 

marked side 

Pachner 

Define $\mu =$ 

$: \; A \otimes A \to A$.

Theorem: For A, c, t the following are equivalent

1. Invariance under local moves holds, c is non-deg., μ has a unit
2. A is a Δ-separable symmetric Frobenius algebra

… state-sum construction

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Here:

- Frobenius: $b : A \otimes A \to k$ invariant non-deg. pairing (dual to c)
- symmetric: b is symmetric
- Δ-separable: $\mu \circ c = 1$ (implies that A-mod is semi-simple)

State-sum construction: spin case

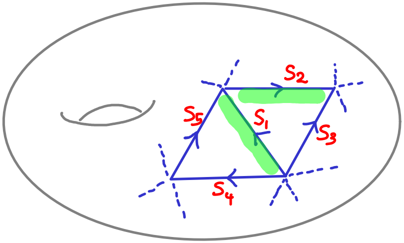
Novak, IR '14

Combinatorial model for spin structures on a surface Σ :

marked triangulation

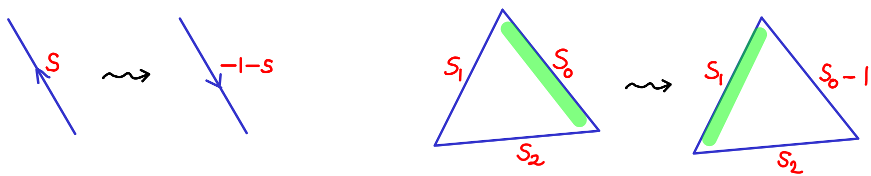
$+$ edge indices $s \in \mathbb{Z}_2$

subject to rule at each vertex



Under of changes of marking:

$(*)$



Can check (by gluing spin triangles) :

$$\left\{ \begin{array}{l} \text{markings on a triangulation} \\ \text{\& edge indices subject to} \\ \text{vertex rule} \end{array} \right\} \Big/ \langle * \rangle \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{spin structures on } \Sigma \text{ up} \\ \text{to isomorphism of spin} \\ \text{structures} \end{array} \right\}$$

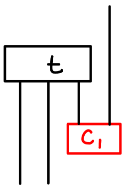
... state-sum construction: spin case

Barrett, Tavares '13,
Novak, IR '14

Data: fin. dim. k-(super)-vector space A , $c_0, c_1 \in A \otimes A$,
$t : A \otimes A \otimes A \to k$.

Choices: marked triang. of Σ , edge indices for spin structure on Σ

Get $Z(\Sigma) = t^{\otimes(\#triang)} \circ ((super)permutation) \circ \bigotimes_{\text{edges } e} c_{s(e)}$

Define $\mu =$ 

$: A \otimes A \to A$.

Theorem: For A, t, c_0, c_1 the following are equivalent

1. Independance under local changes (marking & triangulation) , c_0, c_1 are non-degenerate, μ has a unit
2. A is a Δ-separable Frobenius algebra whose Nakayama automorphism N satisfies $N^2 = id$

… state-sum construction: spin case

Theorem: For A, t, c_0, c_1 the following are equivalent

1. Independance …
2. A is a Δ-separable Frobenius algebra whose Nakayama automorphism N satisfies $N^2 = id$.

Nakayama automorphism:

$N : A \to A$ unique s.t. $b \circ (id_A \otimes N) = b \circ \sigma_{A,A}$

b: non-degenerate invariant pairing on A

$\sigma_{A,A}$: tensor flip in (super) vector sp.

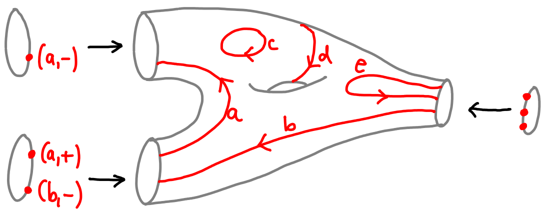
Comments:

- A is symmetric iff $N = id$
- "less conditions on A traded for more geometric structure"
- generalises to r-spin, where $N^r = id$ Novak '15

QFT with defects

Category of bordisms with topological, 1-dimensional defects:

- fix a set of defect conditions D
- objects: disjoint unions of circles and marked points with collars
- morphisms: $\{$ oriented compact surfaces with metric, parametrised boundaries and embedded 1-dimensional sub-manifolds labelled by D $\}/$ $\{$ isometries and isotopies of defect lines rel boundary $\}$



$a, b, \ldots \in D$

A *QFT with defects* Q is a symmetric monoidal functor from the category of bordisms with defects to a suitable category of topological vector spaces.

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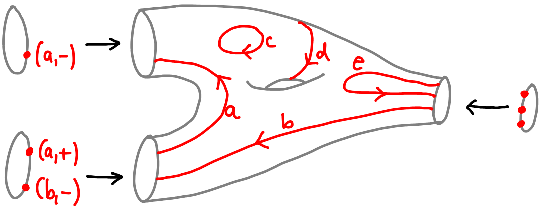

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A *QFT with defects* Q is a symmetric monoidal functor from the category of bordisms with defects to a suitable category of topological vector spaces.

Problem: Currently no non-topological examples known.

There is one example (without defects) if one adds "spin": free fermions (Tener '14).

. . . QFT with defects

Vafa '87, Huang, Kong '06

Way out: Different approach used in *conformal* quantum field theory, similar in spirit but more technical

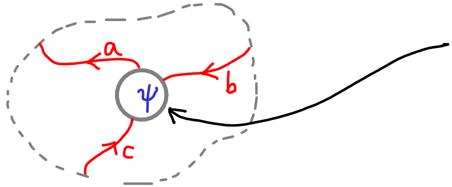
(restrict to genus 0,1, glue spheres along local coordinates of punctures) .

Fröhlich, Fuchs, Schweigert, IR '06
Fjelstad, Fuchs, Stigner '12

Benefit: Many examples from *rational conformal field theory*

State sum inside a QFT with defects: Orbifolds

Need defect junctions:



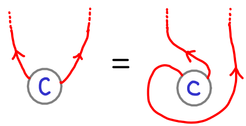
state space $\mathcal{H}$ assigned to circle with marked points

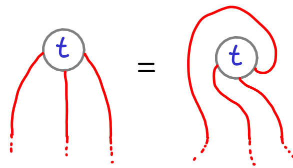
junction: $\psi \in \mathcal{H}$

Topological junction: invariant under isotopies moving boundary component labelled by ψ

Data: defect $A \in D$, topological junctions c, t , subject to conditions as in state sum construction:

- invariance under change of marking





- invariance under Pachner moves

Get: new theory Q_{orb} from old QFT Q with defects by setting

$$Q_{\mathrm{orb}}(\Sigma) \;:=\; Q\left(\begin{array}{l} \Sigma \text{ with defect network} \\ \text{constructed from } A, c, t \end{array} \right)$$

... orbifolds

Davydov, Kong, IR '11

Observation: A QFT Q with defect set D gives rise to a *rigid* *tensor category of defect conditions* $\mathcal{D}(Q)$

- objects: lists of defect conditions, i.e. of elements of D
- morphisms: topological junction fields
- tensor product: concatenation of lists

As in state sum construction on finds:

If A is a Δ-separable symmetric Frobenius algebra in $\mathcal{D}(Q)$, the resulting c, t satisfy the required invariance conditions.

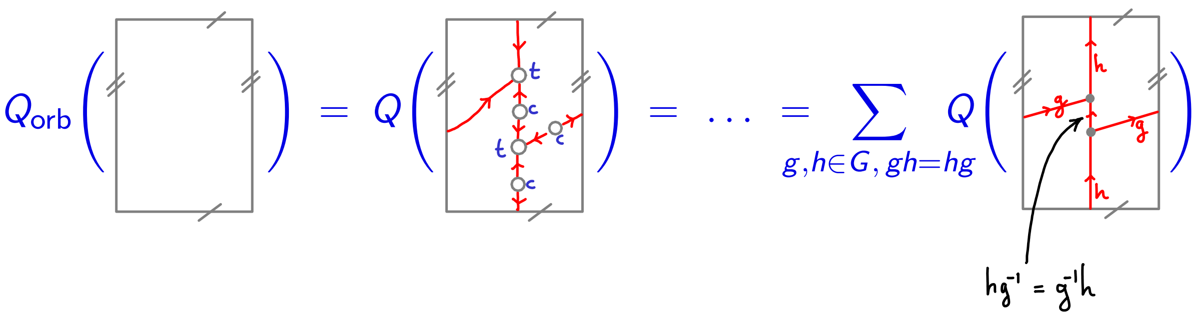
Why is this called orbifold?

A faithful tensor functor $\iota : G \to \mathcal{D}(Q)$ from a group (with only unit morphisms) to $\mathcal{D}(Q)$ describes a G-symmetry of Q .

Suppose $\mathcal{D}(Q)$ has direct sums. Set $A = \bigoplus_{g \in G} \iota(g)$.
One checks: A is a Δ-separable Frobenius algebra.

If A is symmetric, one can construct the orbifold theory Q_{orb} .

Example: Evaluate Q_{orb} on a torus.



Application to rational CFT

Fröhlich, Fuchs, Schweigert, IR '09

Theorem:

Let V be a rational vertex operator algebra. Every CFT C whose left and right symmetry contains V and which has a unique vacuum state can be written as an orbifold (not necessarily of group-type) of any other such CFT D :

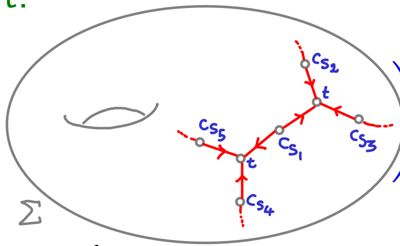
There exists Δ-separable symmetric Frobenius algebra in $\mathcal{D}(D)$ such that $C = D_{\text{orb}}$.

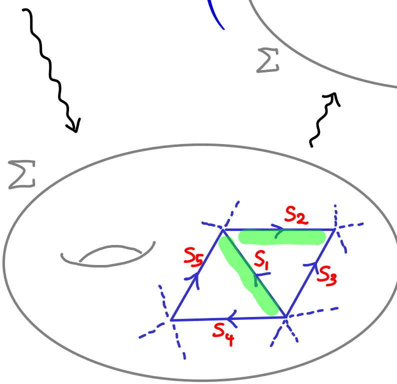
State sum inside a QFT with defects: Spin

Novak, IR (in prep.)

Result:

Let Q be a QFT with defects. A Δ-separable Frobenius algebra in $\mathcal{D}(Q)$ with $N^2 = id$ produces junctions c_0, c_1, t satisfying the required invariance conditions s.t.

$Q_{\text{spin}}(\Sigma_{\text{spin}}) \;:=\; Q\Big($  $\Big)$



is well-defined.

Also in spin case: explicit examples from rational CFT.

Summary

Carry out state sum construction of

- 2d oriented TFT
- 2d spin TFT

inside a QFT with defects to obtain

- (generalised) orbifold QFTs
- spin QFTs