Continuous probabilities, random points, Bernoulli's theorems, and geometric probability applications

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Rules to compute probabilities

- Discrete case – classical probability definition – the probability that an element $\mathbf{X}$ randomly chosen from a finite set Ω belongs to the subset $\mathbf{A} \subseteq \Omega$ is given by

$$\mathbb{P}(\mathbf{X} \in \mathbf{A} | \mathbf{X} \in \Omega) = \frac{\#\mathbf{A}}{\#\Omega}, \quad \mathbf{A} \subset \Omega, \quad \#\Omega < \infty.$$

- Continuous case – geometrical probability definition – the probability that a point $\mathbf{X}$ randomly chosen in a region Ω lies in a subregion $\mathbf{B} \subseteq \Omega$ is given by

$$\mathbb{P}(\mathbf{X} \in \mathbf{B} | \mathbf{X} \in \Omega) = \frac{\mu(\mathbf{B})}{\mu(\Omega)}, \quad \mathbf{B} \subseteq \Omega, \quad \mu(\Omega) < \infty,$$

where μ denotes some measure of the given regions (area, volume, ...)

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where μ denotes some measure of the given regions (area, volume, ...)

- Nevertheless, in the use of these definitions it is assumed
  - $\mu(\Omega) < \infty$ ($\#\Omega < \infty$) a finite measure of the universe; and
  - Equipossibility/Equiprobability (i.e., equal probability of the basic events).
    - Symmetry;
    - **Principle of insufficient reason of Bernoulli and Laplace** – if we have no reason to believe that one way will occur preferentially compared to another, then the events will occur equally likely.

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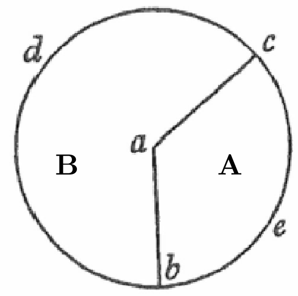
Isaac Newton (written 1664/1666, published in 1976)



- A ball of negligible size falls perpendicularly upon the centre of a horizontal circle divided into two unequal sectors **A** and **B**. Suppose that the ratio of areas of these sectors is 2 to $\sqrt{5}$.
- And if the ball falls in the sector **A** the player wins a, and if it falls into sector **B** wins b.
- Newton claims that the "hopes" of the player worth

$$\frac{2a + b\sqrt{5}}{2 + \sqrt{5}}.$$

- Newton shows that a chance of a simple event can be irrational, and
- discover the basic rule for the geometric probability definition (chance is proportional to the area).

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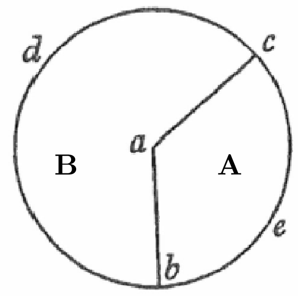

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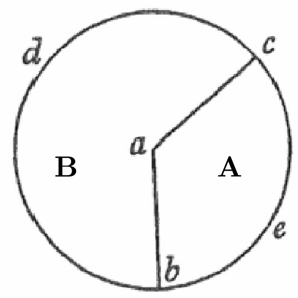

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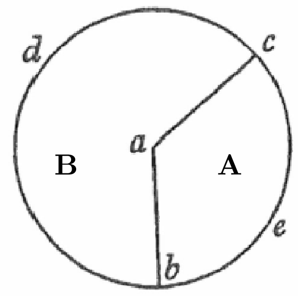

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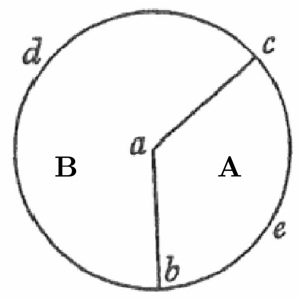

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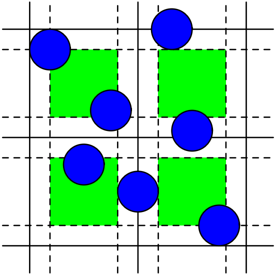
Buffon's problems — French Royal Academy of Science
Presented in 1733 and published in 1777

First problem — Le jeu du Franc-carreau



- A round coin is tossed at random on a large plane area covered by regular tiles (squares in 1733; triangles, rhombi and hexagons in 1777) and one of the players bets that the coin hits only one tile (while the other bets that it hits more of them).
- Buffon solves the problem noting that it is sufficient to use the place where the center of the coin drops on the pavement.
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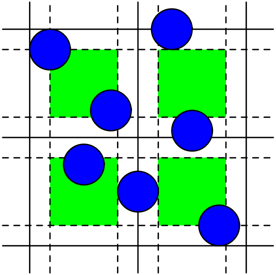

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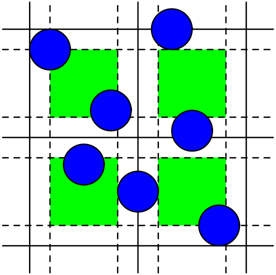

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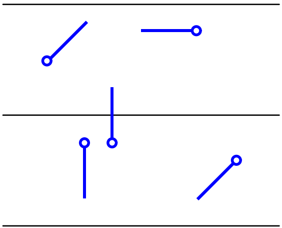
The second problem – Buffon's needle problem



- In a room, the floor of which is simply divided by parallel joints, a needle is thrown in the air, and one of the players bets that the needle will not cross any of the lines, while the other bets that it will cross.
- Using integral calculus, Buffon obtained $\mathbb{P} = \frac{2\ell}{\pi\, d}$, where ℓ is the needle length and d the distance between the parallels (for $d > \ell$).
- Buffon didn't solve the case $d \leq \ell$, only provided (an erroneous) solution (which Laplace corrects in 1812).
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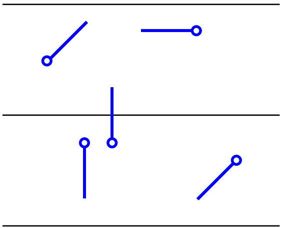

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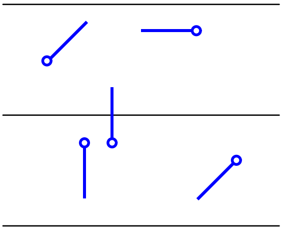

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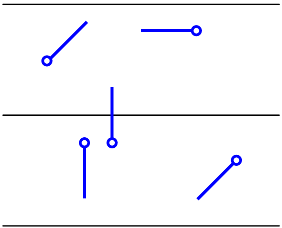

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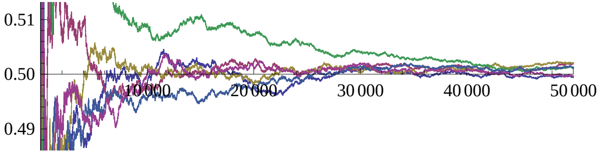
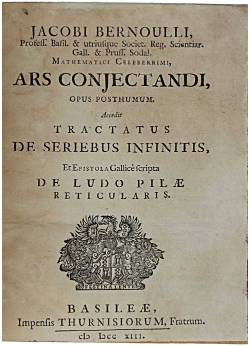
Jacob Bernoulli (1655–1705)
1713 – *Ars conjectandi*

- **First theorem of stochastic convergence** (Weak Law of Large Numbers)

$$\forall \varepsilon > 0 : \lim_{n \to \infty} \mathbb{P}\left[|\hat{p} - p| < \varepsilon\right] = 1.$$





→ Connecting link between the probability theory and reality (freedom from their dependence on gambling).

- Thus, in n random throws in a region Ω, we have

$$\lim_{n\to\infty} \frac{\mathbf{B}_{hits}}{n} = \mathbb{P}(\mathbf{X} \in \mathbf{B} | \mathbf{X} \in \Omega) = \frac{\mu(\mathbf{B})}{\mu(\Omega)},$$

where $\mathbf{B}_{hits}$ denotes the number of hits on region $\mathbf{B}$.

- Therefore, carrying out (simulating) the problem of Buffon, we get

$$\frac{\mathbf{B}_{hits}}{n} \to \frac{2\ell}{\pi\, d} \Longleftrightarrow \pi \approx \frac{2\ell\, n}{\mathbf{B}_{hits}\, d}, \quad \text{for large n.}$$

allowing calculate experimentally the value of π.

1850 Rudolf Wolf performed 5000 throws.
1855 Augustus de Morgan refers that Ambrose Smith performed 3204 throws;
1864 Asaph Hall refers that O. C. Fox performed several sequences with more than 500 throws;
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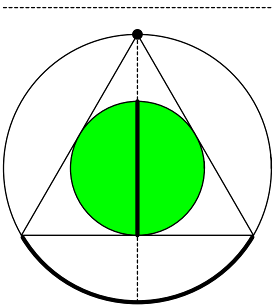

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- **S2** — If he suppose a fix direction and chose, at random, one point in the diameter which is perpendicular $\rightarrow \mathbb{P}_2 = \frac{1}{2}$;
- **S3** — If he chose, at random, one point in the circle and consider the chord that have this point as middle point $\rightarrow \mathbb{P}_3 = \frac{1}{4}$.

There is an one to one relation between the chosen point and the chord.

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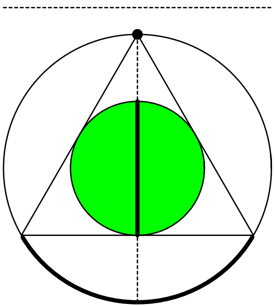

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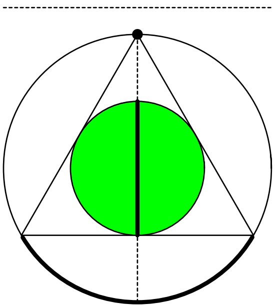

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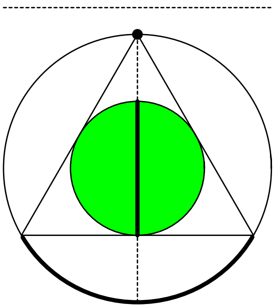

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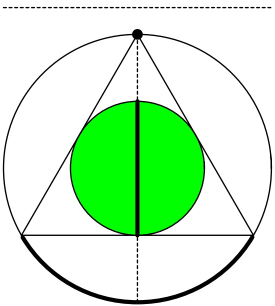

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Probability circa 1914

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- The basic concepts of Probability Theory are not clear
  - *random choice* is a vague concept, having no clear meaning by itself.

- Hilbert (1900) proposed 23 open problems to guide the mathematical investigation during XX century.
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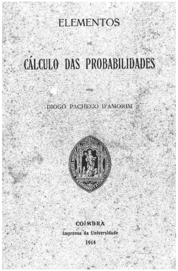
Diogo Pacheco d'Amorim (1888–1976)

PACHECO D'AMORIM, D. (1914). *Elements of Probability Calculus*, Ph.D. Thesis, Coimbra University.

English translation available on http://www.estg.ipleiria.pt/∼rui.santos, done by S. Mendonça, D. Pestana and R. Santos.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ELEMENTS

OF

PROBABILITY CALCULUS

by

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32 *Elementos de Cálculo das Probabilidades*

Dão-se N+1 urnas, contendo a primeira N bolas pretas, a segunda uma bola branca e N−1 pretas, a terceira 2 bolas brancas e N−2 pretas, etc., a última N bolas brancas.

Tira-se uma urna, à sorte, e fazem-se dela $m+n$ tiragens (tornando a pôr na urna cada bola, antes de tirar a seguinte) que dão m bolas brancas e n pretas. Pergunta-se: qual a probabilidade de que a tiragem de ordem $m+n+1$, dê uma bola branca?

*

O problema é resolvido, portanto, pela fórmula do corolário antecedente; pondo

$$p_i = \frac{N-\alpha}{N} \quad \text{e} \quad q_i = \frac{\alpha}{N},$$

virá

$$P = \frac{\sum_{\alpha=0}^{N} \left(\frac{N-\alpha}{N}\right)^{m+1} \left(\frac{\alpha}{N}\right)^{n}}{\sum_{\alpha=0}^{N} \left(\frac{N-\alpha}{N}\right)^{m} \left(\frac{\alpha}{N}\right)^{n}},$$

ou, aproximadamente,

$$P = \frac{\int_0^N \left(\frac{N-\alpha}{N}\right)^{m+1} \left(\frac{\alpha}{N}\right)^{n} d\alpha}{\int_0^N \left(\frac{N-\alpha}{N}\right)^{m} \left(\frac{\alpha}{N}\right)^{n} d\alpha}$$

ou, pondo

$$\alpha = N x,$$

32 *Elements of Probability Calculus*

There are $N+1$ urns, one of them with N black balls, another one with 1 white and $N-1$ black balls, another one with 2 white and $N-2$ black balls, etc., until the last urn, containing N white balls.

Performing $m+n$ extractions of one ball from a randomly chosen urn (always returning the extracted ball to the urn before proceeding to the next extraction), white ball is observed in m occasions, and black ball in n occasions. What is the probability of extracting white ball in the $(m+n+1)$-th extraction?

*

The solution is given in the corollary above, where we may use

$$p_k = \frac{k}{N} \quad \text{and} \quad q_k = \frac{N-k}{N},$$

obtaining

$$\mathbb{P} = \frac{\sum_{k=0}^{N} \left(\frac{k}{N}\right)^{m+1} \left(\frac{N-k}{N}\right)^{n}}{\sum_{k=0}^{N} \left(\frac{k}{N}\right)^{m} \left(\frac{N-k}{N}\right)^{n}},$$

which may be approximated by

$$\mathbb{P} \approx \frac{\int_0^N \left(\frac{\alpha}{N}\right)^{m+1} \left(\frac{N-\alpha}{N}\right)^{n} d\alpha}{\int_0^N \left(\frac{\alpha}{N}\right)^{m} \left(\frac{N-\alpha}{N}\right)^{n} d\alpha}.$$

Using the substitution

$$\alpha = N x,$$

Standard model

Selecting, at random, one element from the sample space

Pacheco d'Amorim considers that this sentence have a clear meaning if:

- **we are the agent** of the selection;
- we possess **full information** about the sample space.

Therefore it can be used as the base to build up Probability Theory.

- CHAPTER I – Discrete case ⇒ random extractions in a finite set ⇒ classical probability definition;
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CHAPTER III – Random figures

Goal: To end up with the paradoxes in geometric probability problems by the definition of random choice of each geometrical figure.

- **Def. 1** – The random choice of an orientation in $\mathbb{R}^n$ is, by definition, the same as randomly throwing a point $(x_1, \ldots, x_n)$ in the set defined by the equation

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 + \cdots + (x_n - x_n')^2 = 1.$$

- **Def. 2** – Randomly throwing a smaller segment on a bigger one is the same as randomly throwing any given point of the smaller segment on the segment it defines when the smaller segment slides over the bigger one.

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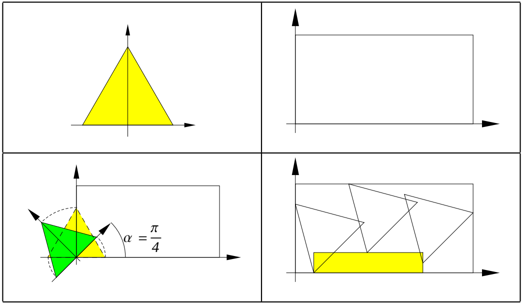
- **Def. 3** – *To throw a straight line at random in a given region A* means, by definition, *to throw a point, at random, in A, and to select at random one direction in the region A*, which determine the straight line.

- $\cdots$

- **Def. 6** – To throw, at random, a plane region in another plane region $\cdots$



- $\cdots$

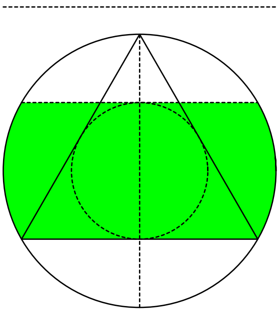
Bertrand's Paradox

- To chose at random one chord in a circle is (by Pacheco d'Amorim's definitions)
  1. to throw a point, at random, in the circle,
  2. to select, at random, one direction.

- The probability is independent of the chord direction (let's assume horizontal);

- The probability will be equal to

$$\mathbb{P} \approx 0.609.$$



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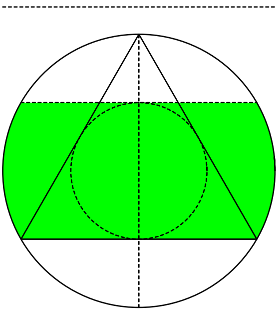

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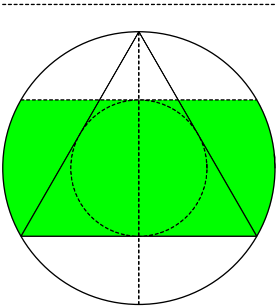
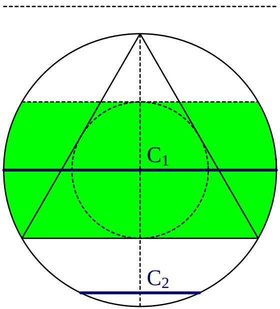
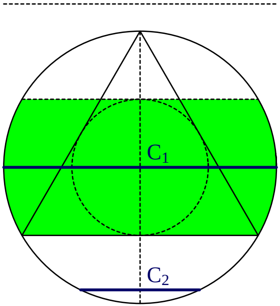




- In Pacheco d'Amorim resolution the possibility of each chord is proportional to its length!
- In fact, all the points that lie in a given chord with the given direction will correspond to the **same** randomly thrown straight line in Pacheco d'Amorim's definition.
- Therefore, we can not consider that the chords are chosen at random (Pacheco d'Amorim has not reached its goal of solving the paradoxes of geometric probability).



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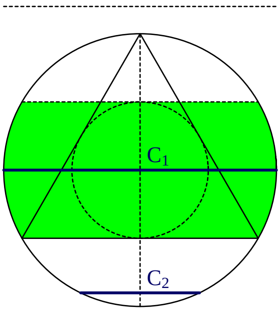

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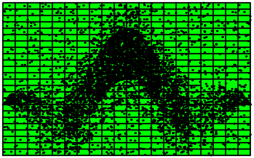
Out of standard model hypothesis
Throws without full information

- **If the support is known, but not the probability law**
  - *Estimate* the probability law.



  - Sub-divided the region $\mathbf{A} = \bigcup_{i=1}^{n} \mathbf{A}_i, \mathbf{A}_i \cap \mathbf{A}_j = \emptyset, \forall i \neq j$;
  - Make a great number of throws in the region, in order to estimate the probability of each region.

- **If we have no information about sample space**
  - We can use the same method, cause the law determination will gives us the regions where the probability is null.

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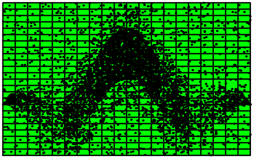

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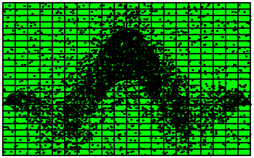

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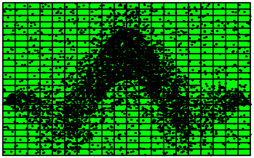

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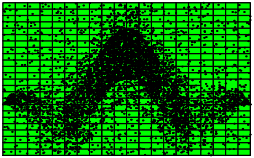

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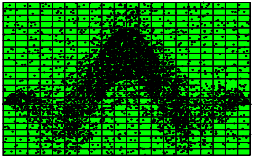

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Is it random?

- **And if we are not the agent?**
  - If the sequence of extractions (throws) are in agreement with Bernoulli's laws, then it can be considered as random.
    - → He analyses if a (long enough) sequence of trials performed by someone else, or even by a mechanical device, can be considered as random.
    - → Thus, it is not required to be a random phenomenon to apply these methodologies (i.e., simulation can be use).
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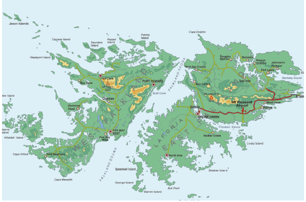
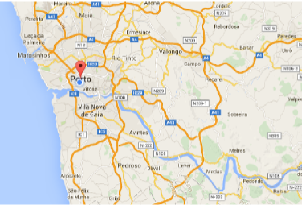
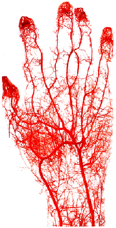
Some current applications of geometric probability



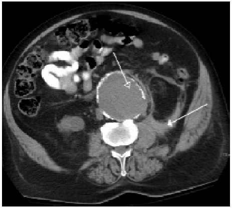
Islands area



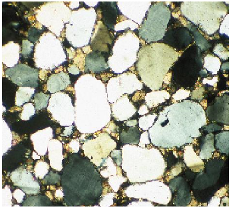
River length



Veins length



Volume of a tumor



% quartz in sandstone

Final Comment

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