The ordinary quiver of the algebra of the monoid of all partial functions on a set

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Monoid representations

- A representation is a (left) module over a $\mathbb{C}$-algebra A.
- A representation of a finite monoid M is a module over the monoid algebra $\mathbb{C}M$:

$$\mathbb{C}M = \{\sum_{i=1}^{|M|} \alpha_i m_i \mid \alpha_i \in \mathbb{C} \quad m_i \in M\}$$

- $\mathbb{C}M$ is a unital associative algebra.

Monoid representations

Given a finite monoid M we want to understand some invariants of the algebra $\mathbb{C}M$:

- Jacobson Radical
- Quiver
- Global dimension

Quiver of an algebra

Definition

The ordinary quiver of a finite dimensional algebra A is a directed graph defined in the following way:

- Vertices: Are in $1-1$ correspondence with the irreducible representations of A (up to isomorphism).
- Arrows: For irreducible representations N_i and N_j the number of arrows from N_i to N_j is

$$\dim e_j(\operatorname{Rad} A/\operatorname{Rad}^2 A)e_i$$

where e_i, e_j are primitive idempotents corresponding to N_i and N_j.

- Equivalently: The number of arrows equals $\dim \operatorname{Ext}^1(N_i, N_j)$.

Quiver of an algebra

Remark

The quiver of an algebra A has no arrows at all if and only if A is a semisimple algebra.

Quiver of monoid algebras

Theorem (Munn-Ponizovski)

Let M be a finite monoid with maximal group H-classes representatives $H_1, \cdots, H_n$ (one for every regular $\mathscr{J}$ class). Then there is a $1-1$ correspondence between the irreducible representations of $\mathbb{C}M$ and those of $\mathbb{C}H_1, \ldots, \mathbb{C}H_n$.

$$\operatorname{Irr} \mathbb{C}M \leftrightarrow \bigsqcup_{k=1}^{n} \operatorname{Irr} \mathbb{C}H_k$$

- *In particular, we can associate to any irreducible representation of M a specific (regular) $\mathscr{J}$ class (called its apex).*

- The apex is the lowest $\mathscr{J}$ class that the representation does not annihilate.

Quiver of monoid algebras

Hence, we can define a partial order on the irreducible representations = the vertices of the quiver.
We say that $N_i \leq N_j$ if $J_i \leq_{\mathscr{J}} J_j$ where J_i, J_j are the corresponding $\mathscr{J}$ classes.

Classical Transformation Monoids

- S_n - Symmetric group. (Permutations on n elements).
- IS_n - Inverse symmetric monoid. (Partial 1-1 maps on n elements).
- T_n - Full Transformations monoid (Functions on n elements)
- PT_n - Partial Transformations monoid (Partial functions on n elements).

Known results

- Putcha (1995): All the arrows in the quiver of $\mathbb{C}\,PT_n$ are going downwards.
- Putcha (1996): Computation of the quiver of $\mathbb{C}\,T_n$ up to $n = 4$ (and some observations for $n > 4$).
- Margolis, Steinberg (2012): Description of the quiver of **DO** monoids (every regular $\mathscr{D}$ class is an orthodox semigroup).
- Steinberg (2015): The quiver of $\mathbb{C}\,T_n$ is acyclic. The global dimension of $\mathbb{C}\,T_n$ is $n - 1$.

Goal

Goal of this talk

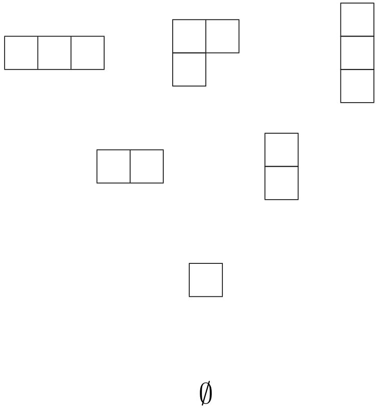
Describing the quiver of $\mathbb{C}PT_n$

Vertices of the quiver of $\mathbb{C}\,\mathrm{PT}_n$

The group $\mathscr{H}$ classes of PT_n are $S_0, \ldots, S_n$ (where $S_0 \cong S_1$) so there is a correspondence between $\operatorname{Irr}\mathrm{PT}_n$ and $\displaystyle\bigsqcup_{k=0}^{n} \operatorname{Irr} S_k$.

Since the irreducible representations of S_k correspond to Young diagrams with k boxes (or partitions of k) we know the the vertices of the quiver correspond to the Young diagrams of with k boxes $0 \leq k \leq n$.

The **vertices** of the quiver of $\mathbb{C}PT_3$:



Isomorphism between $\mathbb{C}\,\mathrm{IS}_n$ and groupoid algebra

Definition

Let G_n be the category whose objects are subsets of $\{1 \ldots n\}$, and whose morphisms are in one-to-one correspondence with elements of IS_n. For every $t \in \mathrm{IS}_n$ there is a morphism $G_n(t)$ from $\operatorname{dom} t$ to $\operatorname{im} t$, multiplication $G_n(s)G_n(t)$ is defined if and only if $\operatorname{im}(t) = \operatorname{dom}(s)$ and the result is $G_n(st)$.

G_n is a groupoid (any morphism is an isomorphism)

Isomorphism between $\mathbb{C}\,\mathrm{IS}_n$ and groupoid algebra

Theorem (Steinberg 2006)

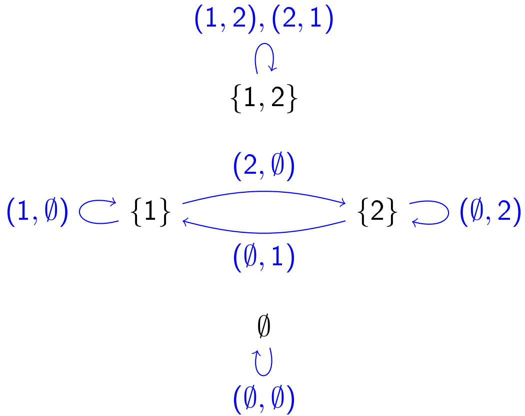
$\mathbb{C}\,\mathrm{IS}_n \cong \mathbb{C}G_n$. Explicit isomorphisms $\varphi : \mathbb{C}\,\mathrm{IS}_n \to \mathbb{C}G_n$ and $\psi : \mathbb{C}G_n \to \mathbb{C}\,\mathrm{IS}_n$ are defined (on basis elements) by:

$$\varphi(s) = \sum_{t \leq s} G_n(t)$$

$$\psi(G_n(s)) = \sum_{t \leq s} \mu(t,s)t$$

where μ is the Mobius function of the natural partial order on IS_n. (Similar argument is true for any inverse semigroup with finite semilattice of idempotents)

$\mathbb{C}\,IS_2$ is isomorphic to the algebra of the category:



Isomorphism between $\mathbb{C}PT_n$ and EI-category algebra

Definition

Let E_n be the category whose objects are the subsets of $\{1 \dots n\}$, and whose morphisms are in one-to-one correspondence with elements of PT_n. For every $t \in PT_n$ there is a morphism $E_n(t)$ from $\operatorname{dom} t$ to $\operatorname{im} t$, multiplication $E_n(s)E_n(t)$ is defined if and only if $\operatorname{im}(t) = \operatorname{dom}(s)$ and the result is $E_n(st)$.

E_n is an EI - category (any endomorphism is an isomorphism)

Proposition

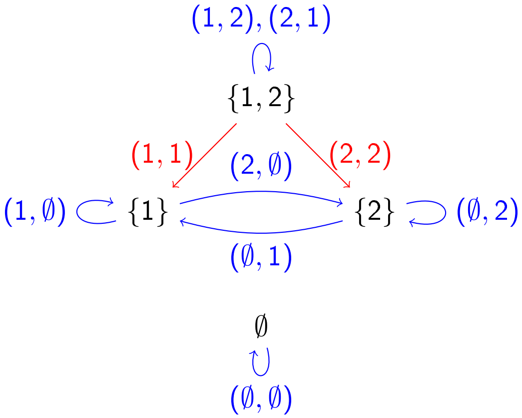
$\mathbb{C}PT_n \cong \mathbb{C}E_n$. *Explicit isomorphisms* $\varphi : \mathbb{C}PT_n \to \mathbb{C}E_n$ *and* $\psi : \mathbb{C}E_n \to \mathbb{C}PT_n$ *are defined (on basis elements) by:*

$$\varphi(s) = \sum_{t \leq s} E_n(t)$$

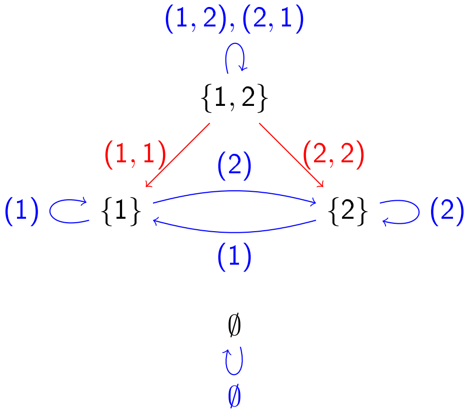
$$\psi(E_n(s)) = \sum_{t \leq s} \mu(t, s)t$$

Isomorphism between $\mathbb{C}PT_n$ and EI-category algebra

$\mathbb{C}PT_2$ is isomorphic to the algebra of the category:



We can regard the morphisms of E_n as being all the **total onto** functions with dom $\subseteq \{1, \ldots, n\}$.

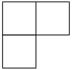
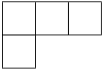


Quiver of $\mathbb{C}PT_n$

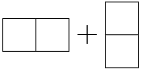
Theorem (IS)

The vertices in the quiver of $\mathbb{C}PT_n$ are in one to one correspondence with Young diagrams with k boxes where $0 \leq k \leq n$. If $\alpha \vdash k$, $\beta \vdash r$ are two Young diagrams such that $r \neq k+1$ then there are no arrows from β to α. If $r = k+1$ then there are arrows from β to α if we can construct β from α by removing one box and then adding two boxes but not in the same column. The number of arrows is the number of different ways that this construction can be carried out.

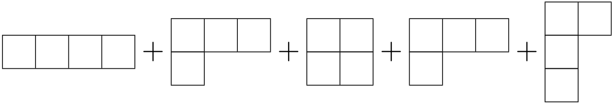
Example

Assume $U =$  and $V =$ .

- Removing one box from U:

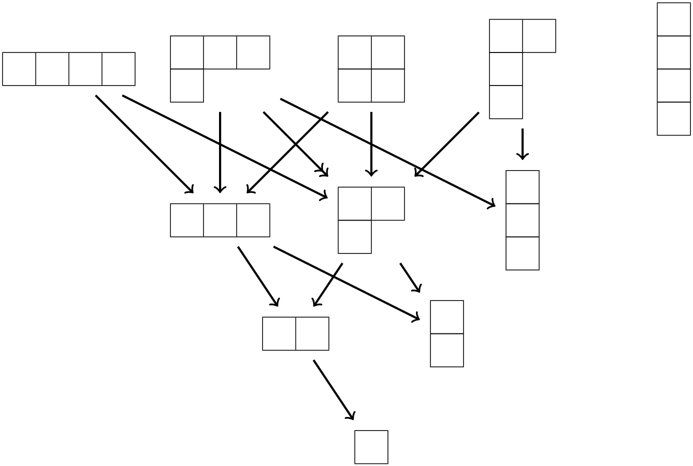


- Adding two, but not in the same column:



So there are two arrows from V to U.

Quiver of $\mathbb{C}PT_4$



$\emptyset$

A generalization: Ehresmann semigroups

A semigroup S with a distinguished set of idempotents $E \subseteq E(S)$ is called Ehresmann if

- E is a subsemilattice of S.
- For every $a \in S$ ther is $e \in E$ such that $ea = a$ (the least such is denoted a^{+})
- For every $a \in S$ ther is $e \in E$ such that $ae = a$ (the least such is denoted a^{*})
- $(ab)^{+} = (ab^{+})^{+}$
- $(ab)^{*} = (a^{*}b)^{*}$

Ehresmann semigroups form a variety.

PT_n is an important example of an Ehresmann semigroup.

A generalization: Ehresmann semigroups

Given an Ehresmann semigroup S with distinguished set of idempotents E, one can define a category $\mathcal{D}$ in the following way:

- Objects: E.
- Morphisms: For every $a \in S$, $\mathcal{D}(a) : a^* \to a^+$

Proposition

If E is finite then $RS \cong R\mathcal{D}$ for any unital ring R.
$\mathcal{D}$ is an EI- category $\Leftrightarrow$ $a^+ = a^*$ implies that a is a group element.

Thank you!