On the Cactus Varieties of Cubic Forms

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Waring rank

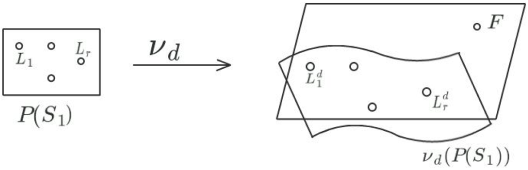
$F \in S_d = K[x_0, \ldots, x_n]_d$ homog. of deg d ($\mathrm{char} K \neq 2, 3$)

Definition

Waring rank: $\min r \in \mathbb{N}$ s.t. $F = L_1^d + \cdots + L_r^d$ with $L_i \in S_1$

Veronese: $\nu_d : \mathbb{P}(S_1) \to \mathbb{P}(S_d), [L] \mapsto [L^d]$

- $\min r \in \mathbb{N}$ s.t. $[F] \in r$-th secant space to $\nu_d(\mathbb{P}(S_1)) := X_{d,n}$
- The shortest length of a smooth finite scheme $\Gamma \subset \mathbb{P}(S_1)$ s.t. $[F] \in \langle \nu_d(\Gamma) \rangle, \Gamma = \{[L_1], \ldots, [L_r]\}$



Cactus rank–Definition

Remove "smooth"

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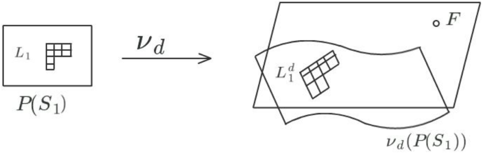

$\mathrm{Sec}_r(X_{d,n}) = \overline{\cup_{\Gamma \in Hilb_r \mathbb{P}(S_1), \Gamma \text{smooth}} \langle \nu_d(\Gamma) \rangle}$ Secant variety to Veronese

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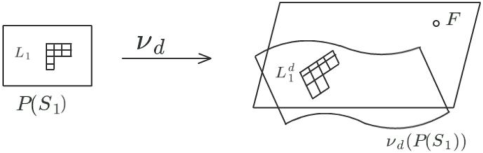

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Cactus rank–History

1999 Iarrobino and Kanev: *scheme length*.

2010 Buczyńska, Buczyński:

- Definition of Cactus variety;
- Study its relation with Secant variety,
- if $d \geq 2r$ and $r \leq i \leq d - r$ the cactus rank $=$ maximal rank of a catalecticant matrix of order i ([B,Brachat, Mourrain]–2014) $=$ *differential length* in [IK], *catalecticant rank* in [BBM].

2014 B, Brachat, Mourrain: catus rank $=$ *generalized rank*.

2012 B, Ranestad: the general cubic form in $n+1$ variables has cactus rank at most $2n+2$.

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Theorem

We focus on **CUBIC** forms

Theorem

When $r \leq 17$, then

$$\dim \mathrm{Cactus}_r(X_{3,n}) = \dim \mathrm{Sec}_r(X_{3,n}).$$

When $18 \leq r \leq 2n+2$, then

$\dim \mathrm{Cactus}_r(X_{3,n}) = \dim W_r(X_{3,n}) > \dim \mathrm{Sec}_r(X_{3,n})$ *and*

$$\dim \mathrm{Cactus}_r(X_{3,n}) =$$

$$= \begin{cases} \min\left\{\frac{1}{48}r^3 - \frac{3}{8}r^2 + rn + \frac{5}{3}r - 2, \binom{n+3}{3} - 1\right\}, \text{if } r \geq 18 \text{ even,} \\ \min\left\{\frac{1}{48}r^3 - \frac{7}{16}r^2 + rn + \frac{119}{48}r - \frac{65}{16}, \binom{n+3}{3} - 1\right\}, \text{if } r \geq 18 \text{ odd.} \end{cases}$$

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Apolarity

$S = K[x_0, \ldots, x_n]$, $T = K[y_0, \ldots, y_n]$

$$y^\alpha\left(x^{[\beta]}\right) = \begin{cases} x^{[\beta-\alpha]} & \text{if } \beta \geq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

S_1 and T_1 are dual spaces.

T is naturally the coordinate ring of $\mathbb{P}(S_1)$.

Definition

$f \in S$. **Apolar ideal:** $f^\perp = \{\varphi \in T \mid \varphi(f) = 0\}$.

$\Gamma \subset \mathbb{P}(S_1)$ is apolar to $F \in S$ if $I_\Gamma \subset F^\perp \subset T$.

Lemma (Apolarity Lemma)

$\Gamma \subset \mathbb{P}(S_1)$ is apolar to $F \in S_d$ iff $[F] \in \langle \nu_d(\Gamma) \rangle \subset \mathbb{P}(S_d)$.

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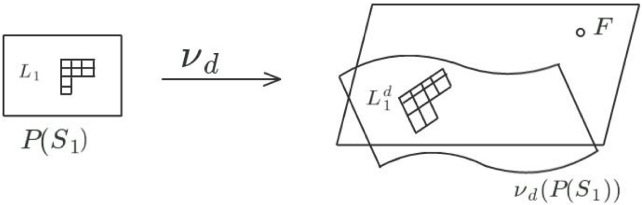

- $\mathrm{crk}(F) = \min r \in \mathbb{N}$ s.t. $\exists \Gamma \in \mathrm{Hilb}_r(\mathbb{P}(S_1)) : [F] \in \langle \nu_d(\Gamma) \rangle$
- $\mathrm{crk}(F) = \min r \in \mathbb{N}$ s.t. $\exists \Gamma \in \mathrm{Hilb}_r(\mathbb{P}(S_1)) : \Gamma$ is apolar to F
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We start with $F \in S_d$, we can compute $F^\perp \subset T$, then we want to describe the minimal $\Gamma \subset \mathrm{Hilb}_r(\mathbb{P}(S_1)) : I_\Gamma \subset F^\perp$.

Local apolar scheme

In order to study the ideals contained in $F^{\perp} \subset T$, the first natural ring to study is $T/F^{\perp}$:

Properties: $T_f := T/f^{\perp}$ local Artinian Gorenstein ring ([IK]):

- Local: The image of T_1 in T_f generates the only max ideal m;
- Artinian: T_f is finitely generated as K-mod;
- Gor: T_f has 1-dim'l socle (the annihilator of the max ideal).

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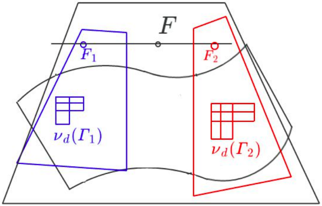
Local apolar scheme

$F \in S_d \Rightarrow \Gamma$ apolar scheme locally Gor. $\Rightarrow$

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_s, \text{ with } \Gamma_i \text{ local A.G.}$$

$$F = F_1 + \cdots + F_s \text{ s.t. } \Gamma_i \text{ apolar to } F_i$$



Local cactus rank

$lcr(F)$ =Local cactus rank of $F = \min_{r\in\mathbb{N}}$ s.t. $\exists$ local $\Gamma \in Hilb_r(\mathbb{P}(S_1))$:
$[F] \in \langle \nu_d(\Gamma) \rangle$

$\Leftrightarrow \Gamma$ is apolar to $F \Leftrightarrow I_\Gamma \subset F^\perp$

$$W_r(X_{d,n}) := \{[F] \in \mathbb{P}(S_d) \,|\, lcr(F) = r\} : r\text{-local cactus variety}$$

$$\mathrm{Cactus}_r(X_{d,n}) = \overline{\{[F] \in \mathbb{P}(S_d) \,|\, cr(F) = r\}} =$$

$$= \overline{\bigcup_{r_1+\cdots+r_s=r} J(W_{r_1}(X_{d,n}), \ldots, W_{r_s}(X_{d,n}))}$$

$\dim W_{r_i}(X_{d,n}) \Rightarrow \dim \mathrm{Cactus}_r(X_{d,n})$

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Local cactus rank

Theorem

If $18 \leq r \leq 2n+2$ *and* $n \geq 8$, *then*

$$\dim \mathrm{Cactus}_r(X_{3,n}) = \dim W_r(X_{3,n})$$

So the cactus rank of a general cubic form can be computed locally.

1. $\dim W_r(X_{d,n}) = ?$
2. How to compute the local cactus rank?

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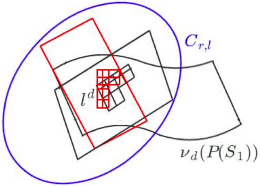
$W_r(X_{d,n})$ the r-local cactus variety

$C_{r,l} = \bigcup_{\mathrm{supp} Z_l = [l^d], l(Z_l) \leq r} \langle Z_l \rangle$



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So we want to study $C_{r,l}$, i.e we want to parameterize the polynomials whose local apolar scheme has given length.

Any Local AG scheme Γ is the AFFINE apolar scheme of a poly $g \in S$ (unique up to a unit in the ring of diff. operators): for any LAG $T^0/I, \exists g \in S^0 s.t. I = g^{\perp}$.

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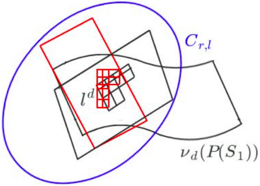

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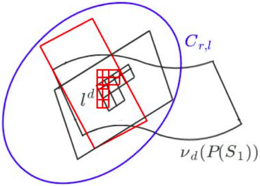

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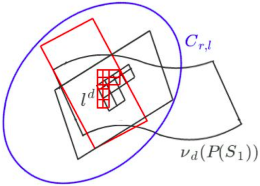

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Which is the link between F having Γ as minimal apolar local scheme and g being the AFFINE polynomial defining Γ?

$$g = \underbrace{g^{(0)} + \cdots + g^{(d)}}_{\deg d \text{ tail of } g} + g^{(d+1)} + \cdots + g^{(l)},\ \deg(g^{(i)}) = i.$$

Proposition

$F \in S_d$, $f = F(1, x_1, \ldots, x_n)$. Let Γ be a scheme of minimal length among local schemes supported at $[l] = [1 : 0 : \ldots : 0]$ that are apolar to $F \Rightarrow \Gamma$ is the AFFINE apolar scheme to a poly $g \in K[x_1, \ldots, x_n]$ whose deg d tail equals f.

So g may be chosen s.t. f is its tail (: Γ is also defined by many g's that does not have f as a tail).

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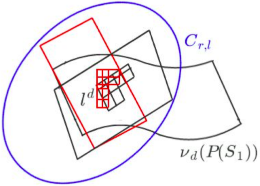
Proof of the Thm

$\mathrm{Cactus}_r(X_{d,n}) = \overline{\bigcup_{r_1+\cdots+r_s=r} J(W_{r_1,n},\ldots,W_{r_s,n})}$

$W_{r,n} = \overline{\bigcup_{l\in S_1} C_{r,l}}$ the r-local cactus variety

$C_{r,l} = \bigcup_{\mathrm{supp} Z_l=[l^d], l(Z_l)\leq r} \langle Z_l \rangle$



Parameterize the set of poly's $g \in K[x_1,\ldots x_n]$ whose affine local apolar scheme has given length. $\overset{\text{Propostion}}{\Longleftrightarrow}$ Parameterize the family of cubic tails $f \in K[x_1,\ldots,x_n]$ of g's.

$C_{r,l}=\{[F]\in\mathbb{P}(S_d) | f = g_{\leq d}$ for some $g \in S_{loc}$ with $\dim \mathit{Diff}(g) \leq r\}$

Proof of the Thm

$C_{r,l} = \bigcup_{\mathrm{supp} Z_l = [l^d], l(Z_l) \le r} \langle Z_l \rangle =$

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Parameterize the family of cubic tails $f \in K[x_1, \dots, x_n]$ of g's.

Find a discrete invariant for LAG schemes, parameterize the cubic tails of all polynomials that define a scheme with given invariant.

The Hilbert function is not good: Ex.:

$g_1 = x^4 + y^3 x + z^2 + t^2$

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They have the same HF:[1 4 3 2 1] but different degree 3 tails

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The HF symmetric decomposition is the good invariant (Iarrobino [Memoirs]):

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Partials of order 0: $g_1 = x^4 + y^3x + z^2 + t^2$

It has deg 4 (so it will contribute to the last column)

It is the only one partial of degree 4 (so we have to put a *1* in the last column)

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\Delta_1 & = & 0 & 0 & 0 & 0 & \\
\Delta_2 & = & 0 & 2 & 0 & &
\end{array}
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Order 0: $g_1 = x^4 + y^3x + z^2 + t^2$, $(0 + 4 = 4 \Rightarrow \Delta_0)$

Order 1: $\partial_x = 4x^3 + y^3$, $\partial_y = 3xy^2$, $(1 + 3 = 4 \Rightarrow \Delta_0)$

$\partial_z = 2z$, $\partial_t = 2t$ $(1 + 1 = 2 \Rightarrow \Delta_2)$

Order 2: $\partial_{xx} = 12x^2$, $\partial_{xy} = 3y^2$, $\partial_{yy} = 6xy$, $(2 + 2 = 4 \Rightarrow \Delta_0)$ $\partial_{zz} = \partial_{tt} = 2$

Order 3: $\partial_{xxx} = 24x \sim \partial_{yyy} = 6x$, $\partial_{xyy} = 6y$ $(3 + 1 = 4 \Rightarrow \Delta_0)$

Order 4: 1 $(4 + 0 = 4 \Rightarrow \Delta_0)$.

Proof of the Thm

$g_1 = x^4 + y^3x + z^2 + t^2$:

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Proof of the Thm

$$g \in S,\ \deg(g) = d,\ g^{\perp} \subset T,\ T_g := T/g^{\perp} \simeq \mathit{Diff}(g) = \{\psi(g) \,|\, \psi \in T\}$$

Iarrobino

T_f is local $\Rightarrow$ one max ideal m.

- m-adic filtration:

  $T_f = m^0 \supset \cdots \supset m^{d+1} = 0$

  $T_f^* = \bigoplus_{i=0}^{d} \frac{m^i}{m^{i+1}}$

- Löewy filtration:

  $T_f = (0 : m^{d+1}) \supset \cdots \supset (0 : m) \supset 0$

Interpr. in terms of partial of f

- $m^i \overset{\tau}{\mapsto}$Partial of order at least i of f (deg $\leq d - i$) (Order of a partial f' of f $=$ largest order of $\psi \in T$ s.t. $f' = \psi(f)$)

- $(0 : m^i) \overset{\tau}{\mapsto}$ $\mathit{Diff}(f)_{i-1}$ $=$partials of deg at most $i - 1$ of f

Proof of the Thm

Order filtration: $\mathit{Diff}(f) = \mathit{Diff}(f)^0 \supseteq \mathit{Diff}(f)^1 \supseteq \cdots \supseteq \mathit{Diff}(f)^d$

Degree filtration: $\mathit{Diff}(f) = \mathit{Diff}(f)_d \supseteq \mathit{Diff}(f)_{d-1} \supseteq \cdots \supseteq \mathit{Diff}(f)_0$

Different filtrations (f not homog)

but

$$\frac{(0 : m^i)}{(0 : m^{i-1})} \cong \left(\frac{m^{i-1}}{m^i}\right)^\vee \qquad\qquad \frac{\mathit{Diff}(f)_{i+1}}{\mathit{Diff}(f)_i} \simeq \frac{\mathit{Diff}(f)^{i+1}}{\mathit{Diff}(f)^i}$$

In particular

$$H_f(i) = \dim_K(\mathit{Diff}(f))_i - \dim_K(\mathit{Diff}(f)_{i-1})$$

has symmetric decomposition:

$$H = \sum_{a \geq 0} \Delta_a$$

each Δ_a symm. around $(d-a)/2$, i.e. $\Delta_a(i) = \Delta_a(d-a-i)$

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$= \{[F]\in\mathbb{P}(S_d) | f = g_{\leq d}$ for some $g\in S_{loc}$ with $\dim \mathit{Diff}(g)\leq r\}$

Found an invariant to stratify poly's with the same degree 3-tail i.e. for poly's with loc. apolar scheme with given length.

Then: Compute the dimension of all the poly's with a given HF decomposition (i.e. of $C_{r,l}$)

$$V(3,\Delta,n) = \{f_{\leq 3}\,|\, f\in K[x_1,\ldots,x_n], \Delta_f = \Delta\}$$

$$C_{r,l} = \bigcup_{l(\Delta)\leq r} V(3,\Delta,n)$$

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Proof of the Thm

Proposition

$r \geq 7$, $v(3, \Delta, n)$ attains its $\max$ for

$$\begin{cases} HF = (1, m-1, m-1, 1), \Delta = (1, m-1, m-1, 1), r = 2m, \\ HF = (1, m-1, m-1, 1, 1), \Delta = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & m-2 & m-2 & 0 & 0 \end{pmatrix}, r = 2m+1 \end{cases}$$

and

$$\dim V(3, \Delta, n) = \begin{cases} M_e := \binom{m+2}{3} + 2m(n-m) + 3m - n - 1, r = 2m, \\ M_o := \binom{m+2}{3} + 2m(n-m) + 3m - 2, r = 2m+1. \end{cases}$$

Proof of the Thm

A general cubic form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ with even local cactus rank $2m$, $m \leq n$ is projectively equivalent to some

$$F = f_3 + x_0 f_2 + x_0^2 f_1 + x_0^3 f_0$$

where

$$\begin{aligned} f_3 &\in \mathbb{C}[x_1, \ldots, x_{m-1}]_3, \\ f_2 &\in \langle x_1, \ldots, x_n \rangle \cdot \langle x_1, \ldots, x_{m-1} \rangle, \\ f_1 &\in \langle x_1, \ldots, x_n \rangle, \\ f_0 &\in \mathbb{C}. \end{aligned}$$

$HF(f) = (1, m-1, m-1, 1)$

The forms of local cactus rank $2n$ form a family of codimension $\binom{n-1}{2} + 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$.

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The forms of local cactus rank $2n+1, n > 3$ form a family of codimension $\binom{n-2}{2} - 1$ in the space of cubic forms $\mathbb{C}[x_0, \ldots, x_n]_3$.

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THANKS!