A Sign-changing solution for an asymptotically linear Schrödinger equation

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The model problem[1]

The propagation of electromagnetic waves through a stratified medium can be can be modelled by a nonlinear Schrödinger equation with saturable nonlinearity given by

$$i\Psi_t + \lambda\Delta_x\Psi + \frac{|\Psi|^2\Psi}{1+s|\Psi|^2} = 0$$

- Ψ denotes the amplitude of the wave,
- s is the saturation parameter of the medium,
- λ is the speed of propagation of the guided wave.

[1] (C. A. Stuart ARMA 1993)

Looking for spatial standing waves $\Psi(t,x) = exp(-i\lambda t)u(x)$

$$\begin{cases} -\Delta u + \lambda u = \dfrac{u^3}{1+su^2} \qquad \text{in} \quad \mathbb{R}^N \ (N \geq 3), \\ \\ \quad u(x) \to 0 \quad \text{if} \quad |x| \to \infty, \end{cases}$$

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- A necessary condition for the existence of nontrivial solutions: $0 < \lambda\, s < 1$ (from Pohozaev identity).

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- For $0 < \lambda\, s < 1$, there is a unique positive radial and radially decreasing least energy solution.[2]

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- A necessary condition for the existence of nontrivial solutions: $0 < \lambda\, s < 1$ (from Pohozaev identity).
- For $0 < \lambda\, s < 1$, there is a unique positive radial and radially decreasing least energy solution.[2]
- The question of interest here is proving that this condition is also sufficient to show the existence of sign-changing solutions.

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The problem

$$-\Delta u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N.$$

(f1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f2) $f(-t) = -f(t)$ for all $t \in \mathbb{R}$;

(f3) $\displaystyle\lim_{t \to 0} \frac{f(t)}{t} = 0$;

(f4) $\exists\, s > 0$; $\displaystyle\lim_{t \to +\infty} \frac{f(t)}{t} = \frac{1}{s}$ and $\displaystyle\frac{f(t)}{t} < \frac{1}{s}$ for all $t \in \mathbb{R}$;

(f5) $\displaystyle\frac{f(t)}{t}$ is an increasing function for all $t > 0$.

(NQ) $$\begin{cases} \displaystyle\lim_{t \to +\infty} [f(t)t - 2F(t)] = +\infty, \quad F(t) = \int_0^t f(\tau)d\tau \\ f(t)t - 2F(t) \geq 0, \ \text{for all } t \in \mathbb{R}, \end{cases}$$

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Example: $f(u) = u^3/(1 + su^2)$

References

- [Nehari, 1961]

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  Nehari introduced a method for finding nodal solution of a superlinear ODE by pasting together positive and negative solutions on alternating annuli.

- [Terracini-Verzini, 2001]

  This method was successfully applied for superlinear ODE's systems to obtain oscillating solutions.

References

- A similar argument was used to obtain sign-changing solutions to radially symmetric PDE's for the superquadratic and subcritical nonlinearities, for example, in

  [Bartsch-Willem, 1993],

  [Bartsch-Weth, 2003],

  [Struwe, 1982]

  [Chabrowski, 1996] (for the critical growth)

References

- Existence result by minimization in a closed set containing all the sign-changing solutions, for example, in

  [Castro-Cossio-Neuberger, 1997] (Superlinear Dirichlet problem)

- [Conti-Terracini-Verzini, 2002] (Superlinear PDE's systems)

Our main result[3]

Theorem

Assume $(f1) - (f5)$ and (NQ) are satisfied. If the parameter $s > 0$, given in condition $(f4)$, satisfies $s \in (0, 1/\lambda)$, then there exists a radial sign-changing solution of

$$-\Delta u + \lambda u = f(u) \quad \textit{in } \mathbb{R}^N$$

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which changes sign exactly once in $\mathbb{R}^N$ and minimises the energy among all possible sign-changing solutions of the problem which are radially symmetric. If this solution is non degenerate and f is in $C^1(\mathbb{R},\mathbb{R})$, then it has Morse index $j\geq N+2$.

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Our contribution

- To apply of the fine construction of [Castro-Cossio-Neuberger][4], differently from them, in an unbounded domain like $\mathbb{R}^N$.
- To face the subtle peculiarities of a nonlinear term which is non homogeneous and asymptotically linear at infinity.

[4] bounded domain in $\mathbb{R}^N$, $f \in C^1$ is superlinear and superquadratic, $f(u) \sim u^p$, $p \in (1, (N+2)/(N-2))$

Difficulties

- In general, the approaches to find nodal solutions of an elliptic equation with a nonlinear term stumble on the fact that the operators $\int_{\mathbb{R}^N}(|\nabla u^{\pm}|^2\,dx$ are not in C^1.

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- In general, the approaches to find nodal solutions of an elliptic equation with a nonlinear term stumble on the fact that the operators $\int_{\mathbb{R}^N}(|\nabla u^{\pm}|^2\, dx$ are not in C^1.

  We are able to avoid this difficulty by recovering the basic ideas used in [Castro-Cossio-Neuberger].
- $(f4)$ states that f does not verify the Ambrosetti-Rabinowitz superquadraticity condition:

$$\theta F(s) \leq f(s)s, \text{ for some } \theta > 2, \; \forall s \in \mathbb{R}.$$

  This inequality is one of the main tools in order to prove the boundedness of the Palais Smale sequence.

The Variational Framework

Let E be the Sobolev space $H^1_{rad}(\mathbb{R}^N)$ of the radial functions with the inner product $\langle u, v\rangle = \int_{\mathbb{R}^N}(\nabla u \nabla v + \lambda uv)\, dx$ and the associated norm by $\|u\| = \langle u, u\rangle^{1/2}$.

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We define $I : E \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda u^2)\,dx - \int_{\mathbb{R}^N} F(u)\,dx.$$

Weak solutions of equation of the equation correspond to critical points of I:

$$I'(u)v = \int_{\mathbb{R}^N}(\nabla u \nabla v + \lambda uv)\,dx - \int_{\mathbb{R}^N} f(u)v\,dx.$$

We follow [Castro-Cossio-Neuberger] in defining the functional $\gamma : E \to \mathbb{R}$ by

$$\gamma(u) = I'(u)u = \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2)\,dx - \int_{\mathbb{R}^N} f(u)u\,dx,$$

and consider the sets

$$\begin{aligned} S &= \{u \in E\backslash\{0\} : \gamma(u) = 0\} \quad \text{Nehari manifold}; \\ \widehat{S} &= \{u \in S : u^+ \not\equiv 0, u^- \not\equiv 0\}; \\ S_1 &= \{u \in \widehat{S} : \gamma(u^+) = 0\}, \end{aligned}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$.

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- We observe that sign-changing solutions of the problem are in S_1

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- Prove that S_1 is a natural constraint, that is, every constrained critical point on S_1 is a free critical point.

S_1 is non empty

Lemma

The set S_1 is non empty.

[5] Bartolo-Benci-Fortunato 1983
[6] Maia-Miyagaki-Soares

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Proof: We start by gluing two solutions of the equation, on in a ball of a fixed radius and the other in an exterior domain, essentially in the way of the Nehari method.

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- There exist respectively $\bar{u} > 0$[5] and $\bar{v} > 0$[6] radial solutions of

$$\begin{cases} -\Delta u + \lambda u &= f(u) \qquad \text{in } B_R(0), \\ u &= 0 \qquad \text{on } \partial B_R(0). \end{cases}$$

$$\begin{cases} -\Delta v + \lambda v &= f(v) \qquad \text{in } \mathbb{R}^N \backslash B_R(0), \\ v &= 0 \qquad \text{on } \partial(\mathbb{R}^N \backslash B_R(0)). \end{cases}$$

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- Define $z = \bar{u} - \bar{v}$
  (thus, $z^+ = \bar{u}$ and $z^- = -\bar{v}$), and fix $\bar{u}$ and $\bar{v}$ as zero outside $B_R(0)$ and $\mathbb{R}^N \backslash B_R(0)$, respectively.

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- define a function g

$$g(t) = \frac{I'(tz)tz}{t^2} = \langle z, z \rangle - \int_{\mathbb{R}^N} \frac{f(tz)}{t} z \, dx, \quad \forall\, t > 0.$$

By (f_3), $g(0) = \lim t \to 0^+ g(t) = \langle z, z \rangle > 0$, and so g is continuous on $[0, \infty)$.

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On the other hand,

$$\lim_{t\to+\infty} g(t) \quad = \quad \langle z, z\rangle - \lim_{t\to+\infty} \int_{\mathbb{R}^N} \frac{f(tz)}{t} z\, dx$$

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$$\begin{aligned} \lim_{t \to +\infty} g(t) &= \langle z, z \rangle - \lim_{t \to +\infty} \int_{\mathbb{R}^N} \frac{f(tz)}{tz} z^2 \, dx \\ &= \langle z, z \rangle - \int_{\mathbb{R}^N} \frac{1}{s} z^2 \, dx \end{aligned}$$

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$$\begin{aligned}\lim_{t\to+\infty} g(t) &= \int_{B_R(0)} \left(\frac{f(\bar{u})}{\bar{u}} - \frac{1}{s}\right) \bar{u}^2\, dx + \int_{R^N\setminus B_R(0)} \left(\frac{f(\bar{v})}{\bar{v}} - \frac{1}{s}\right) \bar{v}^2\, dx \\ &< 0\end{aligned}$$

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  $w^+ \in S? \quad w^- \in S?$

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- Combining $aw^+ + bw^- \neq 0$ with $\gamma(aw^+ + bw^-) = 0$, $aw^+ \not\equiv 0$, $bw^- \not\equiv 0$ and $\gamma((aw^+ + bw^-)^+) = \gamma(aw^+) = 0$

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Hence, S_1 is non empty.

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Proof:

- $S_1 = \gamma^{-1}\{0\} \cap (\gamma \circ h)^{-1}\{0\} \cap \{u \in E : u^+ \neq 0, u^- \neq 0\}$, where $h : E \to E$, $h(u) = u^+$.
- if $2 < q < 2^*$ and $\gamma(u) = 0$ then $|u|_{L^q} > \rho > 0$
- $E = H^1_{rad}(\mathbb{R}^N)$ is compactly imbedded in $L^q(\mathbb{R}^N)$

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Proof: Let $u_c \in S_1$ such that $I(u_c) = c = \inf_{S_1} I$. We have to prove that $I'(u_c) = 0$.

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Let $D = [\frac{1}{2}, \frac{3}{2}] \times [\frac{1}{2}, \frac{3}{2}]$ and $\phi(\xi, \tau) = \xi u^+ + \tau u^-$.

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For $\epsilon > 0$ sufficiently small, there exists $\eta \in C([0,1] \times E, E)$ such that

$$\max_{(\xi,\tau)\in D} I(\eta(1, \phi(\xi, \tau)) < c \qquad (\star)$$

Claim:

$$\eta(1, \phi(D)) \cap S_1 \neq \emptyset \qquad (\star\star)$$

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From $(\star)$ and $(\star\star)$, $\inf_{S_1} I < c$, a contradiction.

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In order to verify $(\star\star)$, define $\varphi(\xi,\tau) = \eta(1, \phi(\xi,\tau))$ and

$$\Psi(\xi,\tau) := (\underbrace{I'(\varphi(\xi,\tau)^+)\varphi(\xi,\tau)^+}_{X(\xi,\tau)}, \underbrace{I'(\varphi(\xi,\tau)^-)\varphi(\xi,\tau)^-}_{Y(\xi,\tau)})$$

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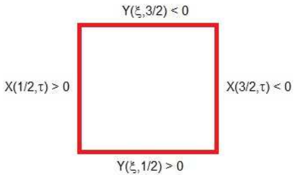
Using that $t \mapsto f(t)/t$ is increasing on $\mathbb{R} \setminus \{0\}$, we have



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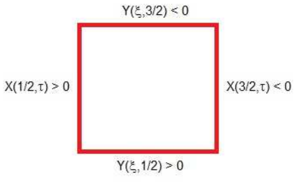

By Miranda's Theorem, there exists $(\xi_0, \tau_0) \in D$ such that $\Psi(\xi_0, \tau_0) = (0, 0)$, as we have claimed

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- $\gamma(u_n^+) = 0$ and $\gamma(u_n^-) = 0 \Rightarrow |u_n^+|_{L^q},\ |u_n^-|_{L^q} \geq \rho > 0$.

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- u changes sign exactly once: as in [Castro-Cossio-Neuberger, 1997].

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Since u is solution, $\gamma(u_A) = \gamma(u_B) = \gamma(u_C) = 0$. Now, $I|_S > 0$ implies

$$I(u_A + u_B) < I(u_A + u_B + u_C) \leq I(u) = c,$$

a contradiction because $u_A + u_B \in S_1$.

Morse index

Since u is radially symmetric, from [Gladiali-Pacella-Weth(2010)] there is $\psi_i \in C^\infty$, with compact support in the open half space $\sum(e_i) = \{x \in \mathbb{R}^N : x \cdot e_i > 0\}$, such that

$$I''(u)(\psi_i, \psi_i) < 0,$$

for any canonical directions e_i, $i = 1, \ldots, N$.

On the other hand, by $(f5)$, $I''(u)(v, v) < 0$ for $v = u^+$ and $v = u^-$.

Using that the support u^+ is in a ball with center at the origin and the support of u^- is the exterior of this ball with the fact that the support of ψ_i is in the open half space $\sum(e_i)$ for every $i = 1, \ldots, N$, we have

$$u^+,\ u^- \notin \mathrm{span}\{\psi_1, \ldots, \psi_{\mathrm{N}}\}$$

and hence we conclude that u has Morse index $j \geq N + 2$.

Open questions

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2. Is there a non-radial positive solution to the problem in the exterior domain $\mathbb{R}^N \setminus B_R(0)$ with Dirichlet boundary condition?
3. Is it possible to find a non radial sign-changing solution by gluing two solutions of the equation, one in a ball of a fixed radius and the other in the exterior of the ball which is positive and non radial?

Open questions

4. Is it possible to solve the previous problems with nonlinear terms in the equation which do not satisfy the condition monotonicity condition: $f(t)/t$ is increasing in $(0, \infty)$?

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   In this case it is not clear that a nontrivial function can be projected on the Nehari manifold associated with the problem. In this case, it may happen that the Pohozaev manifold should be a natural constraint to manage the problem.

Thank you!