How to compute the Stanley depth of a module

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Porto, 11 June 2015

Notations

1. $\mathbb{K}$ is a field;
2. $R = \mathbb{K}[X_1, \ldots, X_n]$ is the standard $\mathbb{Z}^n$-graded polynomial ring;
3. $M = \oplus M_{\mathbf{a}}$ is a finitely generated $\mathbb{Z}^n$-graded R-module;
4. $\preceq$ is the componentwise order on $\mathbb{Z}^n$;
5. We set
$$[\mathbf{a}, \mathbf{b}] := \{\mathbf{c} \in \mathbb{Z}^n \ : \ \mathbf{a} \preceq \mathbf{c} \preceq \mathbf{b}\}$$
with $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$.

Stanley and Hilbert decompositions

Definition

1. A Stanley decomposition of M is a finite family $(R_i, m_i)_{i\in I}$, in which all $m_i \in M$ are homogeneous and R_i are subalgebras of R generated by a subset of the indeterminates of R (for each $i \in I$), such that $R_i \cap \operatorname{Ann} m_i = 0$ for each $i \in I$, and

$$M = \bigoplus_{i\in I} m_i R_i \tag{1}$$

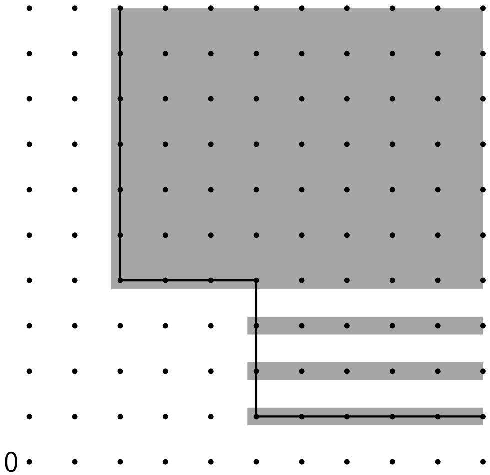
as a multigraded $\mathbb{K}$-vector space.

2. A Hilbert decomposition of M is a finite family $(R_i, \mathbf{s}_i)_{i\in I}$, where $\mathbf{s}_i \in \mathbb{Z}^n$ and R_i are as above (for each $i \in I$), such that

$$M \cong \bigoplus_{i\in I} R_i(-\mathbf{s}_i) \tag{2}$$

as a multigraded $\mathbb{K}$-vector space.

Stanley (Hilbert) decomposition – Monomial ideal



Stanley decomposition of the ideal $\mathfrak{I} = (X^5Y, X^2Y^4) \subset \mathbb{K}[X, Y]$

Stanley (Hilbert) decomposition – Modern art

Stanley (Hilbert) decomposition – Pyrite crystal

Stanley (Hilbert) decomposition – Volleyball ball

Stanley and Hilbert depth

Every Stanley decomposition $(R_i, m_i)_{i \in I}$ of M gives rise to the Hilbert decomposition $(R_i, \deg m_i)_{i \in I}$. We say that a Hilbert decomposition is induced by a Stanley decomposition if it arises in this way. In general, the R-module structure of a Stanley decomposition is different from that of M. Hilbert decompositions depend only on the Hilbert series of M, i.e. they do not take the R-module structure of M into account.

Definition

The depth of a Stanley (resp. Hilbert) decomposition is the minimal dimension of the subalgebras R_i in the decomposition. Equivalently, it is the depth of the right-hand side of (1) (resp. (2)), considered as R-module. The Stanley depth (resp. the Hilbert depth) of M is the maximal depth of a Stanley (resp. Hilbert) decomposition of M. We write $\operatorname{sdepth} M$ (resp. $\operatorname{hdepth} M$) for the Stanley (resp. Hilbert) depth.

A first question

The following question has been raised by Herzog:

Question (Herzog 2013, Question 1.65)

Does there exist an algorithm to compute the Stanley depth of finitely generated $\mathbb{Z}^n$-graded R-modules?

In the particular cases of R-modules which are either monomial ideals $\mathfrak{I} \subset R$, or quotients thereof, this question has been answered in [Herzog, Vladoiu, Zheng 2009].

A second question

A key remark is that, in the cases studied by Herzog, Vladoiu and Zheng, the Hilbert series already determines the module structure. So, the Stanley depth may be computed directly from the Hilbert series of M. Moreover, in these cases

$$\operatorname{sdepth} M = \operatorname{hdepth} M.$$

In fact, the method of [Herzog, Vladoiu and Zheng 2009] extends directly to an algorithm for computing the Hilbert depth of finitely generated multigraded R-modules, studied in [I., Moyano 2014] and [I., Zarojanu 2014]. This leads to the following:

Question

Which Hilbert decompositions are induced by Stanley decompositions?

The answer

A precise answer to the second question implies an answer to the first question.

Fix a Hilbert decomposition $\mathfrak{D} = (R_i, \mathbf{s}_i)_{i \in I}$ of M. We may assume that both M and $\mathfrak{D}$ are (positively) $\mathbf{g}$-determined for some $\mathbf{g} \in \mathbb{N}^n$. Further, we construct a collection of matrices $(A_{\mathbf{a}})_{\mathbf{a} \in [\mathbf{0}, \mathbf{g}]}$, such that all the entries of $A_{\mathbf{a}}$ are linear polynomials totaly determined by $\mathfrak{D}$.

We give an effective criterion to decide whether a given Hilbert decomposition is induced by a Stanley decomposition.

The result

Theorem

Denote by $|\mathbb{K}|$ the cardinality of the field $\mathbb{K}$. The following holds:

(a) *Assume $|\mathbb{K}| = \infty$. Then the given Hilbert decomposition of M is induced by a Stanley decomposition if and only if the determinant of $A_{\mathbf{a}}$ is not the zero polynomial for all $\mathbf{a} \in [\mathbf{0}, \mathbf{g}]$.*

(b) *Assume $|\mathbb{K}| = q < \infty$. Let $P := \prod_{\mathbf{a} \in [\mathbf{0}, \mathbf{g}]} \det A_{\mathbf{a}}$. Let further $\tilde{P}$ be the polynomial obtained from P as follows: From every exponent of every monomial in P, subtract $q - 1$ until the remainder is less than q. Then the given Hilbert decomposition of M is induced by a Stanley decomposition if and only if $\tilde{P} \neq 0$.*

This leads directly to an algorithm for the computation of the Stanley depth of a finitely generated $\mathbb{Z}^n$-graded R-module M.

A third question

As an application of this result, we were able to construct a counterexample which gives a negative answer to the following open question, also raised by Herzog:

Question (Herzog 2013, Question 1.63)

Let M be a finitely generated multigraded R-module with syzygy module Z_k for $k = 1, 2, \ldots$. Is it true that

$$\operatorname{sdepth} Z_{k+1} \geq \operatorname{sdepth} Z_k?$$

Let $n = 6$, $R = \mathbb{K}[X_1, \ldots, X_6]$, $\mathfrak{m}_6$ be the maximal ideal of R and $M = \mathfrak{m}_6 \oplus R^9$. We have computed that

$$\operatorname{sdepth} M = 5 > 4 \geq \operatorname{sdepth} \operatorname{Syz}^1_R(M).$$

Sources

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