Lax pairs and Riemann-Hilbert problems for conjugate conductivity equations

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A pair of conjugate elliptic equations

Conductivity equation for a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$:

$$\mathrm{Div}(\sigma \nabla u) = \partial_x(\sigma \partial_x u) + \partial_y(\sigma \partial_y u) = 0,$$

with a positive conductivity $\sigma(x, y)$. The differential form

$$-(\sigma \partial_y u)dx + (\sigma \partial_x u)dy$$

is closed. If Ω is simply connected, then there is a function v such that

$$\partial_x v = -\sigma \partial_y u, \qquad \partial_y v = \sigma \partial_x u,$$

The form

$$\sigma^{-1} \partial_y v dx - \sigma^{-1} \partial_x v dy$$

is closed and v satisfies

$$\mathrm{Div}(\sigma^{-1} \nabla v) = 0,$$

so that u and v are "σ-conjugate" functions.

conductivity of type $\sigma(x,y) = x^p$, $p \in \mathbb{Z}$

The pair of conjugate equations becomes

$$\Delta u + \frac{p}{x}\partial_x u = 0 \quad \text{and} \quad \Delta u - \frac{p}{x}\partial_x u = 0. \tag{1}$$

We restrict ourselves to Ω a subset of the right-half plane $\mathbb{H}$.

If $U(x_1, \ldots, x_{n+2}) = u(r, x_{n+2})$, $r^2 = x_1^2 + \cdots + x_{n+1}^2$, then

$$\Delta U(x_1, \ldots, x_{n+2}) = \Delta u(r, x_{n+2}) + \frac{n}{r}\partial_r u,$$

whence the name of "Generalized axially symmetric potential theory" given by Weinstein for the study of equations (1).
Specific values of p correspond to various problems in physics.
Classical Methods : For Ω with simple geometry and appropriate choice of coordinates, the method of separation of variables give bases of solutions.
For instance, if $\Omega = \mathbb{D}_a = D(a,1)$, $a > 1$, one may use bipolar coordinates and toroidal harmonics (i.e. Legendre functions of half-integer degrees).

A different approach : Lax pairs and Riemann-Hilbert problems

We follow the unified transform method (cf. A.S. Fokas and coauth.) to solve the boundary value problem

$$\Delta u + \frac{p}{x}\partial_x u = 0, \quad u_t, u_n \text{ given on } \partial\Omega, \quad \Omega = D(a,1), \quad a > 1.$$

Lax pair : a pair of o.d.e. which are compatible iff the equat. is satisfied.
We use complex variables :

$$u_{z\overline{z}} + \frac{p}{2(z+\overline{z})}(u_z + u_{\overline{z}}) = 0,$$

and try to rewrite it in the form $(f(z,\overline{z})u_{\overline{z}})_z + (g(z,\overline{z})u_z)_{\overline{z}} = 0$.
After some computations, one finds possible choice for f and g :

$$\left((k+\overline{z})^{p/2-1}(k-z)^{p/2}u_{\overline{z}}\right)_z - \left((k+\overline{z})^{p/2}(k-z)^{p/2-1}u_z\right)_{\overline{z}} = 0,$$

where $k \in \mathbb{C}$ is a new parameter (the spectral parameter).

Lax Pair :
$$\psi_z(z,k) = (k+\overline{z})^{p/2}(k-z)^{p/2-1}u_z(z),$$
$$\psi_{\overline{z}}(z,k) = (k+\overline{z})^{p/2-1}(k-z)^{p/2}u_{\overline{z}}(z),$$

Closed form and a function Φ of the spectral parameter k

We get that the following differential form is closed in Ω,

$$z \mapsto W(z,k) = \big[(k-z)(k+\overline{z})\big]^{p/2-1}\big[(k+\overline{z})u_z(z)dz + (k-z)u_{\overline{z}}(z)d\overline{z}\big].$$

Depending on the sign and parity of $p \in \mathbb{Z}$, $W(z,k)$ may have a pole or a branch point at k and $-\overline{k}$, but at least we have the global relation :

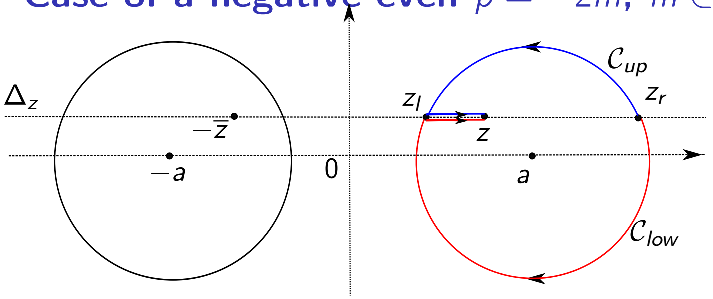
$$\int_\gamma W(z,k) = 0, \qquad k \text{ outside of } \Omega \text{ and } -\overline{\Omega}.$$

We use $W(z,k)$ to define a function of $k \in \mathbb{C}$, depending on z,

$$k \to \phi(z,k) = \int_{z_r}^{z} W(z',k).$$

We consider the case $p < 0$ and distinguish even and odd values :
p even : the function ϕ is defined on $\mathbb{C}$
p odd : the function ϕ is defined on a two-sheeted Riemann surface $\mathcal{S}_z$

Case of a negative even $p = -2m$, $m \in \mathbb{N}$



Paths of integration from z_r to z, respectively along $\mathcal{C}_{up}$ and $\mathcal{C}_{low}$

The function $\phi(z,k)$ has jumps $J(z,k) := \phi^+(z,k) - \phi^-(z,k)$,

For $k \in (z, z_r) \cup (-\overline{z}_r, -\overline{z})$, $\quad J(z,k) = -\int_{\mathcal{C}_a} W(z',k)$, $\quad$ which is known,

$\phi(z,k)$ has poles of order m at $\{z, z_r, -\overline{z}, -\overline{z}_r\}$,

$\phi(z,k)$ behaves at infinity like $k^{-2m-1}(u(z) - u(z_r))$.

Case of a negative even $p = -2m$, $m \in \mathbb{N}$

We renormalize the problem at infinity by defining

$$\widetilde{\phi}(z,k) = ((k-z)(k+\overline{z}))^m \phi(z,k),$$

whose jumps $\widetilde{J}$ are also known.
Let $\widetilde{\phi}_{z_r,-\overline{z}_r}(z,k)$ be the polar part of $\widetilde{\phi}(z,k)$ at z_r and $-\overline{z_r}$, we get

$$\widetilde{\phi}(z,k) - \widetilde{\phi}_{z_r,-\overline{z}_r}(z,k) = \frac{1}{2i\pi}\int_{(-\overline{z}_r,-\overline{z})\cup(z,z_r)} \frac{\widetilde{J}(z,k')}{k'-k}dk',$$

(Plemelj formula) and by letting $k \to \infty$,

$$\begin{aligned} u(z) - u(z_r) &= a_r + a_{-r} - \frac{1}{2i\pi}\int_{(-\overline{z}_r,-\overline{z})\cup(z,z_r)} \widetilde{J}(z,k')dk' \\ &= 2\,\mathrm{Re}(a_r) - \frac{1}{\pi}\,\mathrm{Im}\int_{(z,z_r)} \widetilde{J}(z,k')dk', \end{aligned}$$

Case of a negative even $p = -2m$, $m \in \mathbb{N}$

For the computation of the residue a_r of $\widetilde{\phi}$ at z_r we need the polar part of ϕ at z_r. We have

$$u_z dz = \frac{1}{2}(u_t + iu_n)ds, \qquad u_{\overline{z}} d\overline{z} = \frac{1}{2}(u_t - iu_n)ds,$$

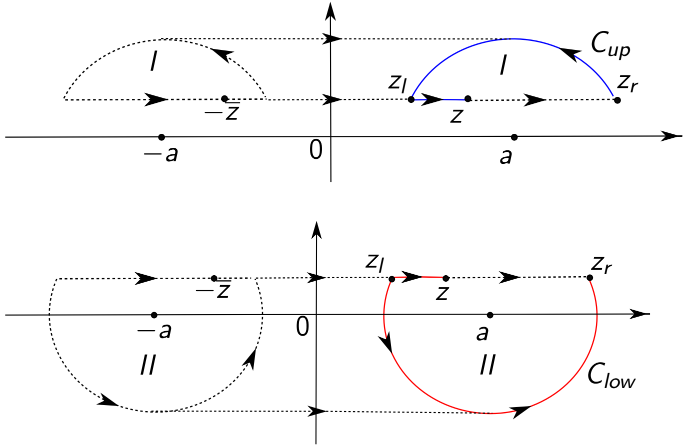
$$\begin{aligned} W(z,k) &= \left((k-z)(k+\overline{z})\right)^{-m-1}\left((k-iy)u_t(z) + ixu_n(z)\right)ds, \\ &= (k-z)^{-m-1} w(z,k) ds. \end{aligned}$$

Then

$$\begin{aligned} \phi(z,k) &= \int_{z_r}^{z} W(z',k) = \frac{1}{m}\int_{z_r}^{z} \partial_{z'}(k-z')^{-m} w(z',k)ds \\ &= \frac{1}{m}\int_{z_r}^{z} \partial_t (k-z')^{-m}\widetilde{w}(z',k)ds, \qquad \widetilde{w}(z',k) = \tau^{-1}(z')w(z',k) \\ &= \frac{1}{m}[(k-z')^{-m}\widetilde{w}(z',k)]_{z_r}^{z} - \frac{1}{m}\int_{z_r}^{z} (k-z')^{-m}\partial_t \widetilde{w}(z',k)ds \end{aligned}$$

Case of a negative odd $p = -2m + 1$, $m \in \mathbb{N}$

$\phi(z, k)$ is defined on a two-sheeted Riemann surface $\mathcal{S}_z$ with a cut $(-\bar{z}, z)$.



Paths of integration on the upper sheet $\mathcal{S}_{z,1}$ (blue), lower sheet $\mathcal{S}_{z,2}$ (red)

The determination of the square root in $W(z, k)$ is chosen in a specific way along the paths of integration.

Case of a negative odd $p = -2m + 1$, $m \in \mathbb{N}$

The function $\phi(z, k)$ has jumps (in terms of the boundary datas) :

– on $\mathcal{C}_{up}$ and $-\overline{\mathcal{C}}_{up}$ on the first sheet $\mathcal{S}_{z,1}$,

– on $\mathcal{C}_{low}$ and $-\overline{\mathcal{C}}_{low}$ on the second sheet $\mathcal{S}_{z,2}$,

$\phi(z, k)$ has poles of order m at $\{z, z_r, -\overline{z}, -\overline{z}_r\}$,

$\phi(z, k)$ behaves like :

$k^{-2m}(u(z) - u(z_r))$ at ∞_1,

$-k^{-2m}(u(z) - u(z_r))$ at ∞_2.

We renormalize the problem at infinity by defining

$$\widetilde{\phi}(z, k) = ((k - z_r)(k + \overline{z}_r))^m \phi(z, k).$$

With $\widetilde{\phi}_{z,-\overline{z}}(z, k)$ the polar part of $\widetilde{\phi}(z, k)$ at z and $-\overline{z}$, the function $\widetilde{\phi}(z, k) - \widetilde{\phi}_{z,-\overline{z}}(z, k)$ is characterized by its jumps and the fact that

$$(\widetilde{\phi} - \widetilde{\phi}_{z,-\overline{z}})(z, \infty_1) = -(\widetilde{\phi} - \widetilde{\phi}_{z,-\overline{z}})(z, \infty_2).$$

Case of a negative odd $p = -2m + 1,\ m \in \mathbb{N}$

We have a Plemelj type formula : with $\lambda(z,k) = \sqrt{(k-z)(k+\bar{z})}$,

$$\widetilde{\phi}(z,k) - \widetilde{\phi}_{z,-\bar{z}}(z,k) = \frac{1}{4i\pi}\int_{\mathcal{C}_{up}\cup-\overline{\mathcal{C}}_{up}} \widetilde{J}(z,k')\left(\frac{\lambda(z,k)}{\lambda_1(z,k')}+1\right)\frac{dk'}{k'-k}$$

$$+\frac{1}{4i\pi}\int_{\mathcal{C}_{low}\cup-\overline{\mathcal{C}}_{low}} \widetilde{J}(z,k')\left(\frac{\lambda(z,k)}{\lambda_2(z,k')}+1\right)\frac{dk'}{k'-k}.$$

Taking the value at ∞_1, we get

$$u(z) - u(z_r) = -\frac{1}{2\pi}\ \mathrm{Im}\int_{\mathcal{C}_a} \frac{\widetilde{J}(z,k)}{\sqrt{(k-z)(k+\bar{z})}}dk.$$

The jump $\widetilde{J}(z,k)$ can be expressed in terms of u_t, u_n, and their tangeant derivatives along $\mathcal{C}_a$.

Dirichlet-Neumann map ($p = -2(m-1)$)

Goal : to reconstruct the map $u_t \to u_n$ only from the "global relation" :

$$\int_{\mathcal{C}_a} [(k-z)(k+\overline{z})]^{-m}((y+ik)u_t(z) - xu_n(z))ds = 0, \quad k \in \mathbb{C} \setminus (\mathcal{D}_a \cup \mathcal{D}_{-a}),$$

or eqvtly, with $f(z) := (x+a)u_n(z+a)$ and chge of var. $\mu := (k+a)^{-1}$

$$\int_{\mathbb{T}} \frac{z^{m-1} f(z) dz}{(1-\varphi(\mu)z)^m (z-\mu)^m} = \text{ known function of } \mu, \quad \mu \in \mathbb{A},$$

where $\quad \varphi(\mu) := \dfrac{-\mu}{1+2a\mu}$ involutive, $\quad \mathbb{A} := \mathbb{D} \setminus D\left(\dfrac{-2a}{4a^2-1}, \dfrac{1}{4a^2-1}\right).$

The goal is to recover f on $\mathbb{T}$. With $f(z) = g_1(z) + \overline{g}_1(1/z)$, we set

$$\Phi_1(\mu) := \int_{\mathbb{T}} \frac{z^{m-1} g_1(z) dz}{(1-\varphi(\mu)z)^m (z-\mu)^m}, \qquad \mu \in \mathbb{A}.$$

Dirichlet-Neumann map

The function $\Phi_1(\mu)$ admits an analytic extension to $\mathbb{D} \setminus \{z_1\}$, where z_1 is the root in $\mathbb{D}$ of

$$1 - \varphi(\mu)\mu = (\mu^2 + 2a\mu + 1)/(1 + 2a\mu).$$

It has a pole of order $2m - 1$ at z_1 and a zero of order m at $-1/2a$. We also have

$$\int_{\mathbb{T}} \frac{z^{m-1} f(z) dz}{(1 - \varphi(\mu)z)^m (z - \mu)^m} = \Phi_1(\mu) - \overline{\Phi}_1(\varphi(\mu))$$

which gives an "almost" decomposition of the left-hand side into analytic and anti-analytic parts. Multiplying our equation by

$$S(\mu) = (\mu^2 + 2a\mu + 1)^{2m-1}/(1 + 2a\mu)^m$$

and taking analytic parts, one obtains

$$S\Phi_1 = S\Psi + P_{2m-2}, \quad \Psi \text{ known function.}$$

Dirichlet-Neumann map

From the formula

$$\Phi_1(\mu) = \frac{2i\pi}{(m-1)!}\left(\frac{h_1(z)}{(1-\varphi(\mu)z)^m}\right)^{(m-1)}(\mu), \quad h_1(z) := z^{m-1}g_1(z),$$

follows that

$$S\Phi_1 = \mathcal{D}_{m-1}(h_1),$$

where $\mathcal{D}_{m-1}$ is a differential operator which has z_1 as a regular singular points. The equation becomes $\mathcal{D}_{m-1}(h_1) = S\Psi + P_{2m-2}$.

Lemma : The linear differential equation of order $m-1$,

$$\mathcal{D}_{m-1}(h) = H, \quad H \text{ analytic near } z_1,$$

has at most one solution $h(z)$, analytic near z_1. Moreover, if the second member $H(z)$ is a polynomial $P_{2m-2}(z)$, then the solution $h(z)$ exists and is also a polynomial of degree at most $2m-2$.

Dirichlet-Neumann map

We have the equation

$$S\Phi_1 = S\Psi + P_{2m-2} \quad \text{i.e.} \quad \mathcal{D}_{m-1}(h_1) = S\Psi + \mathcal{D}_{m-1}(\widetilde{P}_{2m-2}),$$

hence

$$h_1 = h + \widetilde{P}_{2m-2},$$

where h unique (and computable) solution of $\mathcal{D}_{m-1}(h) = S\Psi$.
Finally, the polynomial $\widetilde{P}_{2m-2}$ can be recovered (by going back to the global relation), hence also h_1, g_1, f and the Neumann data u_n.

For positive even coefficient p, one has to use another Lax pair.

For the inverse map Neumann $\to$ Dirichlet, one uses the conjugate equation (which consists in changing p into $-p$).