Generic properties of random subgroups of free groups, and random group presentations

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- Today, we concentrate on free groups, but there are also results on subgroups of hyperbolic groups, on 1-relator groups, etc. (Gilman, Kapovich, Miasnikov, Ollivier, Osin, Sapir, Schupp, Shpilrain, Špakulová, ...)

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- if $d < \frac{1}{2}$, G is generically infinite and hyperbolic, and if $d > \frac{1}{2}$, G is generically degenerate (trivial or $\mathbb{Z}/2\mathbb{Z}$)

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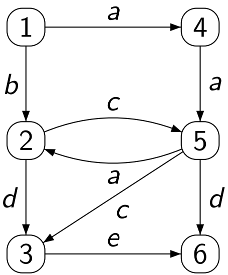

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- If the central tree property holds, then $\vec{h}$ freely generates H

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- if no word of length $\frac{1}{2}(\min \vec{h} - 2\text{lcp}(\vec{h}))$ occurs twice as a factor of the elements of $\vec{h}$ and $\vec{h}^{-1}$, then $\langle \vec{h} \rangle$ is malnormal

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- [Arzhantseva, Ol'shanskii, Jutsikawa] If we fix k and pick a k-tuple $\vec{h}$ of reduced words of length at most n uniformly at random, then exponentially generically, $\langle \vec{h} \rangle$ satisfies the central tree property and is malnormal

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- [Bassino, Nicaud, W] The result actually holds for larger tuples. If we pick a $|B_n|^d$-tuple $\vec{h}$ of reduced words of length at most n uniformly at random, then exponentially generically $\langle \vec{h} \rangle$ has the central tree property if $d < \frac{1}{4}$, and is malnormal if $d < \frac{1}{8}$

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- We consider the following generalization. . .

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- Examples: the uniform distribution on words of length n, the density model and the few-generator or few-relator models fall under this model

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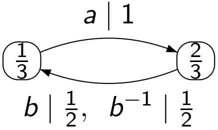
- Suppose in addition that $\liminf \mathbb{R}_n(C_n) > 0$, and let $\lambda < \frac{1}{2}$ and $d < \frac{\lambda}{2}$. Draw an α^{-dn}-tuple $\vec{h}$ of cyclically reduced words of length at most n according to $(\mathbb{R}_n)_n$. Then, exponentially generically, $\vec{h}$ has small cancellation property $C'(\lambda)$

- 

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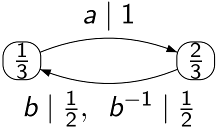
- 

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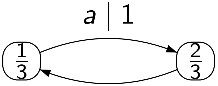

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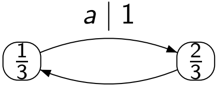

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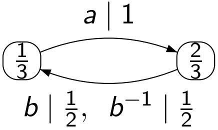
- 

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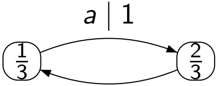

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- Up to α-density $\frac{1}{4}$ (resp. $\frac{1}{8}$), a tuple of reduced words of length at most n generates a subgroup with the central tree property (resp. a malnormal subgroup)

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- In the Markovian automaton for the uniform distribution, $\pi_{[2]} = \frac{1}{2|A|-1}$, and we retrieve the classical results

Thank you for your attention!