Hopf algebroids belong in commutative algebra

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Outline

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1^{st} part

Construction of topological spaces

There is a very flexible procedure to build new topological spaces, namely, passing to the quotient by an equivalence relation.

Let X be a topological space and $\sim$ an equivalence relation on X.
Its quotient topological space is usually denoted $X/{\sim}$.

Let $\Gamma_\sim \subset X \times X$ be the graph of the equivalence relation. There are two canonical maps (induced by projections)

$$\Gamma_\sim \rightrightarrows X$$

Notice that $X \to X/{\sim}$ is the coequalizer of this diagram

$$\Gamma_\sim \rightrightarrows X \longrightarrow X/{\sim}$$

Schemes

Let us recall the notion of scheme: it is a **ringed space** $(X, \mathcal{O}_X)$ that locally resembles an **affine scheme** $(\mathrm{Spec}(A), \widetilde{A})$.

> *For every commutative ring A we associate a topological space, $\mathrm{Spec}(A)$, formed by the prime ideals of A —the* ***points*** *in the geometry of A— endowed with the Zariski topology.*
> *There is a canonical defined sheaf of rings $\widetilde{A}$ over $\mathrm{Spec}(A)$, that enjoys two properties.*
>
> 1. *The global sections of the sheaf recover the ring A.*
> 2. *The stalk of the sheaf $\widetilde{A}$ at the point corresponding to a prime $\mathfrak{p}$, is the local ring $A_{\mathfrak{p}}$, the localization of A with respect to the elements that do not belong to $\mathfrak{p}$.*

It has turned out that the category of schemes is rich enough to express (and solve) the classical problems of algebraic geometry and it has vastly expanded its scope of applications encompassing, for instance, questions in algebraic number theory.

The problem of constructing schemes

The category of schemes has nice properties e.g. it has finite limits.

In particular it has fibered squares, a construction that underlies the philosophy of base change, essential in modern philosophy of algebraic geometry.

However it lacks finite colimits, therefore one cannot use coequalizers to construct schemes with specified properties.

How can one try to specify properties of a scheme and prove its existence?

Yoneda's lemma

Grothendieck's initial idea was to consider Yoneda's lemma.

Let C be a category and Set the category of sets. There is a fully faithful embedding

$$h\colon \mathsf{C} \longrightarrow \mathsf{Fun}(\mathsf{C}^{\mathsf{o}}, \mathsf{Set})$$

that sends an object $X \in \mathsf{C}$ to the contravariant functor

$$h_X := \mathrm{Hom}_{\mathsf{C}}(-, X)$$

and on maps is defined by postcomposition.

The category $\mathsf{Fun}(\mathsf{C}^{\mathsf{o}}, \mathsf{Set})$ has all sorts of limits (inherited from Set).

Statement of the representability problem

Let $\mathsf{C} = \mathsf{Sch}$ be the category of schemes (over a base).

In view of the previous discussion, one may try to specify a functor $F \in \mathsf{Fun}(\mathsf{Sch}^{\circ}, \mathsf{Set})$ and afterwards find a scheme X such that $F \cong h_X$, this is called the problem of representability of F.

At some point, there was a hope to have a nice representability theorem, i.e. a list specifying certain natural conditions on the functor F that guarantees the existence of a scheme X such that $F \cong h_X$.

Failure of representability

Unfortunately the representability question is very difficult.

There is no collection of natural conditions on a functor $F \in \mathsf{Fun}(\mathsf{Sch}^{\mathsf{o}}, \mathsf{Set})$ that ensures its representability by a scheme. The situation is cured if we restrict ourselves to the category of projective schemes, where the existence of global homogeneous rings of coordinates allow for a lot of geometrical meaningful constructions.

One problem that arises is that, contrary to the differential context, the inverse function theorem does not hold, in other words a *differential isomorphism* may not be a *local isomorphism*, and this does not allow patching maneuvers that are natural in the topological setting.

In classification problems the functors we need to represent take values in categories in some sense more general than Set.

2^{nd} part

What if a coequalizer exists?

Let us postulate the existence of a coequalizer

$$X' \rightrightarrows X \longrightarrow \mathbf{X}$$

As $\mathbf{X}$ is obtained by a certain twisted patching on X, if we want the result to be quasi-compact (a natural enough assumption in algebraic geometry), we have to take for X an affine scheme, say $X = \operatorname{Spec}(A_0)$.

Let us look for a more canonical replacement for X'. Take $R := X \times_{\mathbf{X}} X \subset X \times X$. We have:

$$R \rightrightarrows X \longrightarrow \mathbf{X}$$

We'll assume further that R is also affine, say $R = \operatorname{Spec}(A_1)$.

The underlying structure

Notice that R is an equivalence relation inside $X \times X$, therefore we have:

$$\begin{array}{c} \overset{\delta}{\curvearrowleft} \\ i \circlearrowleft R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \longrightarrow \mathbf{X} \end{array}$$

1. the two projections p_1, p_2,
2. the map of identities δ (reflexivity),
3. the interchanging of factors i (symmetry),
4. the composition of pairs m (transitivity)

$$\begin{array}{ccc} R \times_X R & \xrightarrow{\quad m \quad} & R \\ \downarrow \wr & & \downarrow \wr \\ X \times_{\mathbf{X}} X \times_{\mathbf{X}} X & \xrightarrow{\quad p_{13} \quad} & X \times_{\mathbf{X}} X \end{array}$$

Unravelling the structure

We see that **X** is morally determined by the data

$$(X, R, p_1, p_2, \delta, m, i).$$

Notice that this is a groupoid object inside the category of affine schemes:

1. X is the scheme of objects.
2. R is the scheme of morphisms.
3. The maps p_1 and p_2 are, respectively, the source and target maps.
4. The map δ assigns the identities.
5. The map m corresponds to the composition of morphisms.
6. The map i assigns an inverse to every morphism, expressing the fact that our internal category is a groupoid.

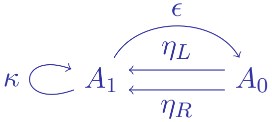
Thinking algebraically

Taking global sections, the tuple that morally determined **X**

$$\mathbf{X} \equiv (X, R, p_1, p_2, \delta, m, i)$$

can be described by the dual looking diagram within the category of commutative rings:

Let, as before, $X = \operatorname{Spec}(A_0)$ and $R = \operatorname{Spec}(A_1)$, the we have



together with

$$\nabla \colon A_1 \longrightarrow A_1 \, {}_{\eta_R}\!\otimes_{\eta_L} A_1$$

corresponding to the composition m.

This constitutes the structure usually called Hopf algebroid.

Hopf algebroids

Let us spell out the structure of Hopf algebroid for $A_\bullet := (A_0, A_1)$:

- it possesses some structure morphisms $\eta_L, \eta_R \colon A_0 \rightrightarrows A_1$,
- counit $\epsilon \colon A_1 \longrightarrow A_0$,
- conjugation $\kappa \colon A_1 \longrightarrow A_1$,
- comultiplication $\nabla \colon A_1 \longrightarrow A_1 \,{}_{\eta_R}\!\otimes_{\eta_L} A_1$;

satisfying the following properties:

1. $\epsilon\eta_L = \mathrm{id}_{A_0} = \epsilon\eta_R$;
2. if $j_1, j_2 \colon A_1 \to A_1 \,{}_{\eta_R}\!\otimes_{\eta_L} A_1$ are defined by $j_1(b) = b \otimes 1$, $j_2(b) = 1 \otimes b$, then $\nabla\eta_L = j_1\eta_L$ and $\nabla\eta_R = j_2\eta_R$.
3. $\kappa\eta_L = \eta_R$ and $\kappa\eta_R = \eta_L$;
4. $(\mathrm{id}_{A_1} \otimes \epsilon)\nabla = \mathrm{id}_{A_1} = (\epsilon \otimes \mathrm{id}_{A_1})\nabla$;
5. $(\mathrm{id}_{A_1} \otimes \nabla)\nabla = (\nabla \otimes \mathrm{id}_{A_1})\nabla$;
6. if μ is multiplication in A_1, then $\mu(\kappa \otimes \mathrm{id}_{A_1})\nabla = \eta_R\epsilon$ and $\mu(\mathrm{id}_{A_1} \otimes \kappa)\nabla = \eta_L\epsilon$;
7. $\kappa\kappa = \mathrm{id}_{A_1}$.

Plain stacks

Let **S** be a category endowed with a Grothendieck topology.

A fibered category over **S** is a functor

$$p_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{S},$$

such that, for $f \colon U \to V$ in **S** and $\eta \in \mathbf{F}$, with $p_{\mathbf{F}}(\eta) = V$, there is a cartesian morphism $\phi \colon \xi \to \eta$ in **F** such that $p_{\mathbf{F}}(\phi) = f$, i.e. $f^*\eta = \xi$.

> *A technical condition guarantees that the comma categories $\mathbf{F}(V) := \mathbf{F} \downarrow V$ are groupoids for every $V \in \mathbf{S}$.*

For a covering $\{f_i \colon U_i \to U\}_{i \in I}$ in **S**, there is a category of descent data $\mathbf{F}(\{f_i \colon U_i \to U\}_{i \in I})$ defined by collections $\{\xi_i \in \mathbf{F}(U_i)\}_{i \in I}$ plus isos

$$\phi_{ij} \colon p_2^*\xi_j \to p_1^*\xi_i \quad \text{in} \quad U_{ij} := U_i \times_U U_j$$

satisfying the cocycle condition and morphisms that respect the underlying structure. There is a canonical map

$$\mathbf{F}(U) \longrightarrow \mathbf{F}(\{f_i \colon U_i \to U\}_{i \in I})$$

Concept of stack

Let $p_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{S}$ be a fibered category (in groupoids) over a site $\mathbf{S}$.

1. $\mathbf{F}$ is a prestack over $\mathbf{S}$ if for every covering $\{f_i \colon U_i \to U\}_{i \in I}$ the functor $\mathbf{F}(U) \longrightarrow \mathbf{F}(\{f_i \colon U_i \to U\}_{i \in I})$ is fully faithfull.
2. $\mathbf{F}$ is a stack over $\mathbf{S}$ if for every covering $\{f_i \colon U_i \to U\}_{i \in I}$ the functor $\mathbf{F}(U) \longrightarrow \mathbf{F}(\{f_i \colon U_i \to U\}_{i \in I})$ is an equivalence of categories.

An affine groupoid scheme $(\mathrm{Spec}(A_0), \mathrm{Spec}(A_1))$ defines a fibered category that its in fact a prestack, denoted $[\mathrm{Spec}(A_0), \mathrm{Spec}(A_1)]'$.

The lack of effectivity of descent data may be solved by a general process called stackification yielding a well defined stack that we will denote as:

$$\mathsf{Stck}(A_\bullet) := [\mathrm{Spec}(A_0), \mathrm{Spec}(A_1)]$$

the stack associated to the Hopf algebroid $A_\bullet = (A_0, A_1)$.

Geometric stacks

The (lax, weak) 2-category of stacks over a site possesses 2-fibered products. The case of interest for us is when **S** is the (big) site over a base scheme S, that for our purposes it might be chosen affine, e.g. $\mathrm{Spec}(\mathbb{Z})$; the Grothendieck topology it's the so-called *étale* topology.

A 1-morphism $\mathbf{F} \to \mathbf{G}$ is called representable (by schemes) if given a scheme X and a 1-morphism $X \to \mathbf{G}$, the stack $\mathbf{F} \times_{\mathbf{G}} X$ is equivalent to one induced by a scheme.

The special properties of a stack of the form $\mathsf{Stck}(A_\bullet)$ are the following

1. It is locally an affine scheme, i.e there is a scheme $X = \mathrm{Spec}(A_0)$ and a smooth and surjective 1-morphism of S-stacks $p\colon X \longrightarrow \mathbf{X}$.
2. The diagonal $\delta\colon \mathbf{X} \longrightarrow \mathbf{X} \times_S \mathbf{X}$ is representable by affine schemes.

These kind of stacks are called geometric stacks and arise quite often in natural constructions in geometry like moduli problems (of curves, abelian varieties, vector bundles,. . .).

Objects, morphisms

Let us discuss briefly the 2-category of (smooth) Hopf algebroids as a generalization of the (ordinary) category of commutative rings.

Objects: Hopf algebroids $A_\bullet$ (with η_L and η_R smooth), as before.

1-morphisms: Let $A_\bullet$ and $B_\bullet$ be two Hopf algebroids,a 1-morphism of Hopf algebroid is a couple of maps

$$\varphi_\bullet \colon A_\bullet \longrightarrow B_\bullet$$

with $\varphi_i \colon A_i \longrightarrow B_i$ homomorphism of rings for $i \in \{1, 2\}$ respecting with all the structural homomorphisms.

2-morphisms

Let there be given two 1-morphisms $\varphi_\bullet$ and $\psi_\bullet$ between the Hopf algebroids $A_\bullet$ and $B_\bullet$, Let $\alpha\colon \varphi_\bullet \Rightarrow \psi_\bullet$ be a 2-morphism.

$$A_\bullet \underset{\psi_\bullet}{\overset{\varphi_\bullet}{\rightrightarrows}} B_\bullet \qquad \alpha \colon \varphi_\bullet \Downarrow \psi_\bullet$$

In explicit terms, α is represented by a homomorphism $\overline{\alpha}\colon A_1 \to B_0$ such that the following diagram commutes

$$\begin{array}{ccc}
A_1 & \xrightarrow{\quad \nabla \quad} & A_1\,{}_{\eta_R}\!\otimes_{\eta_L} A_1 \\
{\scriptstyle \nabla}\big\downarrow & & \big\downarrow{\scriptstyle \psi_1 \otimes \eta_L\overline{\alpha}} \\
A_1\,{}_{\eta_R}\!\otimes_{\eta_L} A_1 & \xrightarrow[\eta_R\overline{\alpha} \otimes \varphi_1]{} & B_1\,{}_{\eta_R}\!\otimes_{\eta_L} B_1
\end{array}$$

Comparison with stacks

Does any morphism of stacks $f\colon \mathbf{X} \to \mathbf{Y}$ induce a morphism of the corresponding Hopf algebroids?

Yes, if we choose an appropriate presentation.

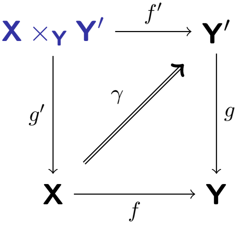
Let $p\colon V \to \mathbf{Y}$ a presentation, i.e. $V = \mathrm{Spec}(A_0)$ and p is smooth and surjective.
Let $V \times_{\mathbf{Y}} V = \mathrm{Spec}(A_1)$, $U := \mathbf{X} \times_{\mathbf{Y}} V = \mathrm{Spec}(B_0)$ and $U \times_{\mathbf{X}} U = \mathrm{Spec}(B_1)$.
The canonical map $q\colon U \to \mathbf{X}$ gives a presentation of $\mathbf{X}$. The choices ensure that $\mathbf{Y} = \mathsf{Stck}(A_\bullet)$, $\mathbf{X} = \mathsf{Stck}(B_\bullet)$ Finally, f induces a homomorphism of Hopf algebroids

$$\varphi_\bullet\colon A_\bullet \longrightarrow B_\bullet$$

such that $f = \mathsf{Stck}(\varphi_\bullet)$

Pull-back squares

Yet another illustration, how to describe a 2-pull-back square?



Choose appropriate presentations giving Hopf algebroids $A_\bullet$, $A'_\bullet$ and $B_\bullet$ such that $\mathbf{Y} = \mathsf{Stck}(A_\bullet)$, $\mathbf{Y}' = \mathsf{Stck}(A'_\bullet)$ and $\mathbf{X} = \mathsf{Stck}(B_\bullet)$.
Then we can write $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Y}' = \mathsf{Stck}(C_\bullet)$ where

$$C_0 = B_0 \otimes_{A_0} A_1 \otimes_{A_0} A'_0 \quad \text{and} \quad C_1 = B_1 \otimes_{A_0} A_1 \otimes_{A_0} A'_1$$

with suitable structure morphisms.

3^{rd} part

The affine dictionary

It is well know the following dictionary

Affine dictionary		
Geometry		*Algebra*
Affine schemes	$\leftrightsquigarrow$	Commutative Rings
Morphisms	$\leftrightsquigarrow$	Homomorphisms
Closed subsets	$\leftrightsquigarrow$	Ideals
Quasi-coherent sheaves	$\leftrightsquigarrow$	Modules
...	$\leftrightsquigarrow$	...

The stacky dictionary

Over a geometric stack $\mathbf{X}$ we consider the site $\mathrm{Aff}_{\mathrm{fppf}}/\mathbf{X}$ formed by affine schemes flat of finite presentation over $\mathbf{X}$.

Using Hopf algebroids we may push the analogy as follows:

Stacky dictionary		
Geometry		*Algebra*
Geometric stacks	$\leftrightsquigarrow$	Hopf algebroids
Morphisms	$\leftrightsquigarrow$	Homomorphisms (up to. . .)
Closed subsets	$\leftrightsquigarrow$	Invariant ideals
Quasi-coherent sheaves	$\leftrightsquigarrow$	Comodules
. . .	$\leftrightsquigarrow$	. . .

A homomorphism of Hopf algebroid gives a 1-morphism between the corresponding stacks, but the other way round requires a special choice of presentation (this can be expressed as a Morita-like issue).

Comodules over a Hopf algebroid

Let $A_\bullet := (A_0, A_1)$ be a Hopf algebroid.
An $A_\bullet$-comodule (M, ψ_M) (on the left) is an A_0-module M together with an A_0-linear map

$$\psi_M \colon M \longrightarrow A_1 \; {}_{\eta_R}\!\otimes_{A_0} M,$$

such that:

1. $(\nabla \otimes \mathrm{id}_M)\psi_M = (\mathrm{id}_{A_1} \otimes \psi_M)\psi_M$ (coassociativity),
2. $(\epsilon \otimes \mathrm{id}_M)\psi_M = \mathrm{id}_M$ (counitarity).

The map ψ_M is called the structure map of the comodule M.

The category of comodules

Theorem

If A_1 is a flat A_0-module (either through η_L or η_R), then the category of $A_\bullet$-comodules, $A_\bullet$-coMod, is a Grothendieck category.

Theorem

Let $\mathbf{X} = \mathsf{Stck}(A_\bullet)$. The categories $\mathsf{Qco}(\mathbf{X})$ and $A_\bullet$-coMod are equivalent.

Corollary

The Abelian category $\mathsf{Qco}(\mathbf{X})$ is a Grothendieck category.

Non functoriality of flat sites

A general 1-morphism of geometric stacks $f\colon \mathbf{X} \to \mathbf{Y}$ does not induce a (continuous) morphism between the sites $\mathrm{Aff}_{\mathrm{fppf}}/\mathbf{Y}$ and $\mathrm{Aff}_{\mathrm{fppf}}/\mathbf{X}$ —unless f itself is an affine morphism, i.e. represented by affine schemes.

Unfortunately, it is neither possible to construct a morphisms of topos between the associated topos $\mathbf{X}_{\mathrm{fppf}}$ and $\mathbf{Y}_{\mathrm{fppf}}$ because topologies finer that the étale rarely posses exact inverse images. That's why some other schools consider big toposes, but then one had to deal with the (in)dependence of universe.

On the bright side, a functorial formalism is still possible if we restrict to sheaves of modules, and, particularly, quasi-coherent sheaves.

Induced adjunction

By employing the usual technique of comparing sites trough a 1-morphism of geometric stacks $f\colon \mathbf{X} \to \mathbf{Y}$ we are able to construct the following

Proposition

Let $\mathcal{F} \in \mathbf{Y}_{\mathsf{fppf}}$. We denote by $f^{-1}\mathcal{F}$ the sheaf associated to the presheaf $f^{\mathsf{p}}\mathcal{F}$. We get a pair of adjoint functors

$$\mathbf{X}_{\mathsf{fppf}} \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathbf{Y}_{\mathsf{fppf}}.$$

Caution: In general f^{-1} is not exact, that's why the couple (f^{-1}, f_*) does not define a topos morphism. Then the problem of how to transport algebraic structures by the functor f^{-1} arises.

Structure transport

Lemma

Let $\mathbf{I}$ be a category that possesses finite products and $F_1, \ldots, F_r$ a family of functors (presheaves) $F_i \colon \mathbf{I}^{\circ} \to \mathsf{Set}$, $i \in \{1 \ldots r\}$. The natural map

$$\varinjlim_{\mathbf{I}} (F_1 \times \cdots \times F_r) \longrightarrow \left(\varinjlim_{\mathbf{I}} F_1\right) \times \cdots \times \left(\varinjlim_{\mathbf{I}} F_r\right)$$

is an isomorphism.

The category used to construct f^{-1} possesses finite products.

Consequence

The functor f^{-1} preserves algebraic structures.

Adjunction

Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of geometric stacks.

Proposition

1. If $\mathcal{F} \in \mathrm{Qco}(\mathbf{Y})$ then $f^*\mathcal{F} \in \mathrm{Qco}(\mathbf{X})$.
2. If $\mathcal{G} \in \mathrm{Qco}(\mathbf{X})$ then $f_*\mathcal{G} \in \mathrm{Qco}(\mathbf{Y})$.

Theorem

There is a pair of adjoint functors

$$\mathrm{Qco}(\mathbf{X}) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Qco}(\mathbf{Y}).$$

2-Functoriality

Proposition

Let $f\colon \mathbf{X} \to \mathbf{Y}$ and $g\colon \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms of geometric stacks, it holds that $(gf)_* = g_* f_*$.

Corollary

Consequently, $(gf)^* \cong f^* g^*$.

Moreover:

Theorem

Let $f_1, f_2\colon \mathbf{X} \to \mathbf{Y}$ be 1-morphisms of geometric stacks, and $\zeta\colon f_1 \Rightarrow f_2$ a 2-morphism. There are isomorphisms

1. $\zeta_*\colon f_{1*} \xrightarrow{\sim} f_{2*}$
2. $\zeta^*\colon f_2^* \xrightarrow{\sim} f_1^*$

2-Functoriality and comodules

Proposition

Let $\varphi\colon B_\bullet \to A_\bullet$ be a homomorphism of Hopf algebroids. There is a pair of adjoint functors $A_0 \otimes_{B_0} - \dashv \mathsf{U}^\varphi$

$$\begin{array}{ccc} X & \xrightarrow{\operatorname{Spec}\varphi_0} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbf{X} & \xrightarrow[f]{} & \mathbf{Y} \end{array}$$

Let us denote by $\Gamma_p^{\mathbf{X}}\colon \mathsf{Qco}(\mathbf{X}) \to A_\bullet$-coMod the equivalence of categories between quasi-coherent sheaves on $\mathbf{X}$ and comodules over $A_\bullet$-coMod.

Theorem

1. There is a natural isomorphism of functors $\Gamma_q^{\mathbf{Y}} f_* \xrightarrow{\sim} \mathsf{U}^\varphi \Gamma_p^{\mathbf{X}}$.
2. There is a natural isomorphism of functors $\Gamma_p^{\mathbf{X}} f^* \xrightarrow{\sim} A_0 \otimes_{B_0} \Gamma_q^{\mathbf{Y}}$.

Geometric features of the adjunction

Some facts that make the categories of comodules look "geometric". Let $\varphi\colon B_\bullet \to A_\bullet$ be a homomorphism of Hopf algebroids.

1. The functor U^φ is not exact, just left exact. Therefore, homomorphisms of Hopf algebroids have cohomology. As a nice property U^φ commutes with coproducts.
2. The functor $A_0 \otimes_{B_0} -$ is right exact, and is exact whenever B_0 is flat over A_0, in which case we say that φ is a flat homomorphism of Hopf algebroids.
3. There is a projection formula

$$(\mathsf{U}^\varphi M) \otimes^{\mathsf{c}}_{A_\bullet} N \cong \mathsf{U}^\varphi(M \otimes^{\mathsf{c}}_{B_\bullet} (A_0 \otimes_{B_0} N))$$

   and similarly flat base change, etc.
4. The category $A_\bullet$-coMod does not have enough projectives unless it is Morita equivalent to a ring category in which case $\mathsf{Stck}(A_\bullet)$ is equivalent to an affine scheme.