Magnetic complex structures

Brian C. Hall AMS-EMS-SPM International Meeting Special Session on Complex Time in Quantum Physics and Geometry

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Credits

- Joint work with **Will Kirwin**.
- References: (1) Hall–Kirwin *Math. Ann.* 2011, (2) Hall-Kirwin *J. Geom. Phys.* 2015.
- Results motivated by ideas of **Thomas Thiemann**, Class. Quantum Gravity 1996.
- **Web site**: www.nd.edu/~bhall/

Polarizations

- A **polarization** on (N, ω) is an **integrable Lagrangian distribution** (possibly complex)
- For each $z \in N$, choose Lagrangian subspace $P_z \subset T_z^{\mathbb{C}}(N)$
- **Real case**: $P_z = \overline{P_z}$ (use Frobenius theorem)
- **Purely complex case**: $P_z \cap \overline{P_z} = \{0\}$ (use Newlander–Nirenberg theorem)

In geometric quantization

- Introduce **line bundle** L with connection over (N, ω) with curvature ω
- Quantum Hilbert space is space of sections that are **covariantly constant** in direction of $\overline{P_z}$
- In purely complex case, get space of **holomorphic sections**

Questions

- How to compare Hilbert spaces with different polarizations (BKS method)?
- Given one real, one complex, can we find unitary **Segal–Bargmann transform** relating two spaces. (Case of $T^*(K)$ with K a compact Lie group.)
- How to construct interesting examples of polarizations?

Thiemann's proposal

- Start with **vertical polarization** P_0 on $N = T^*(M)$
- Push forward by a **Hamiltonian flow** Φ_t, yielding P_t
- Try to analytically continue to pure **imaginary** value of t
- Hope to get family of purely **complex** polarizations

Interpreting imaginary time

- Start with function f constant along leaves of vertical polarization:

$$f = g \circ \pi$$

- Compose with Φ_t:

$$g \circ \pi \circ \Phi_t$$

(constant along leaves of P_{-t})

- Try to analytically continue to $t = i\tau$:

$$g \circ \pi \circ \Phi_{i\tau}$$

(constant along leaves of $P_{-i\tau} = \overline{P_{i\tau}}$)

Proposal

Problem

Given Hamiltonian flow Φ_t*, can we find complex structure* J_τ *(*$\tau > 0$*) on (open subset) of* $T^*(M)$ *such that* **holomorphic functions** *are precisely functions of the form*

$$g \circ \pi \circ \Phi_{i\tau}$$

where $g \circ \pi \circ \Phi_{i\tau}(z)$ *is defined by analytic continuation of* $t \mapsto g \circ \pi \circ \Phi_t(z)$.

Simple example

- Flow of $p^2/2$ on $T^*(\mathbb{R})$:

$$\Phi_t(x, p) = (x + tp, p)$$

- Take $g(x) = x^n$, so

$$g \circ \pi \circ \Phi_t(x, p) = (x + tp)^n$$

- Putting $t = i\tau$ gives

$$(x + i\tau p)^n$$

  These are holomorphic w.r.t. complex structure defined by $z = x + i\tau p$, $\tau > 0$.

Geodesic flow

- Let (M, g) be a (compact) real-analytic Riemannian manifold
- Let $E(x, p) = \frac{1}{2}g(p, p)$ be the energy function, with Φ_t the associated **geodesic flow** on $T^*(M)$
- Let P_t be the push-forward of vertical polarization by Φ_t

First result

Theorem (H–Kirwin, 2011)

One can analytically continue P_t *to positive imaginary time* $t = i\tau$, *on a neighborhood* U *of the zero-section in* $T^*(M)$. *The result is a* **Kähler structure** *on* $U \subset T^*(M)$.

When $\tau = 1$, *the result is the* **adapted complex structure** *previously defined by Lempert–Szőke and Guillemin–Stenzel.*

The holomorphic functions on U *can be computed as in Thiemann's proposal as*

$$F(z) = g \circ \pi \circ \Phi_{i\tau}(z)$$

for some function g *on* M.

Methods of analytic continuation

1. Let $P_z(t)$ be the polarization from pushing forward vertical by Φ_t. Then analytically continue the map $t \mapsto P_z(t)$ as a map into the **Grassmannian of complex Lagrangian subspaces**.
2. Embed M into complex manfold X as in Bruhat–Whitney. Then embed $U \subset T^*(M)$ into X by the map

$$\Lambda(x, p) = \exp_x(i\tau p)$$

   via analytic continuation of **exponential map** $p \mapsto \exp_x(p)$. [cf. Szőke, *Math. Ann.* 1991]
3. Given real-analytic function g on M and point $z \in T^*(M)$, analytically continue the map

$$t \mapsto g \circ \pi \circ \Phi_t(z)$$

Additional results

- Can naturally understand the Guillemin–Stenzel **Kähler potential**

$$\kappa(x, p) = |p|^2$$

  from this perspective, by finding polarized sections of relevant line bundles
- **Jacobi field** perspective also arises naturally, by finding vectors in P_t and then analytically continuing

Sphere case

- Define **quadric**

$$Q^n = \left\{ \mathbf{z} \in \mathbb{C}^n \middle| z_1^2 + \cdots + z_n^2 = 1 \right\} \quad \text{(no complex conjugates!)}.$$

- Then $T^*(S^n)$ has global adapted complex structure, and $T^*(S^n)$ is biholomorphic to Q^n
- First identify $T^*(S^n)$ with $T(S^n) = \{(\mathbf{x}, \mathbf{p}) |\ |\mathbf{x}| = 1,\ \mathbf{x} \cdot \mathbf{p} = 0\}$
- Then $f : T(S^n) \to Q^n$:

$$f(\mathbf{x}, \mathbf{p}) = \cosh |\mathbf{p}|\ \mathbf{x} + i \frac{\sinh |\mathbf{p}|}{|\mathbf{p}|}\ \mathbf{p}$$

The magnetic flow

- Assume that (M, g) has **closed 2-form** β, interpreted as a **magnetic field**
- Consider symplectic form on $T^*(M)$ given by

$$\omega^\beta := \omega - \pi^*(\beta)$$

  where ω is the canonical 2-form on $T^*(M)$
- Energy function is (still) $E(x, p) = \frac{1}{2}g(p, p)$
- **Magnetic flow** is the Hamiltonian flow of E with respect to ω^β.

Results on magnetic flow

Setting: (M, g) compact, real-analytic, with β closed, real-analytic 2-form on M. Let Φ_t^β be the magnetic flow.

Theorem (H–Kirwin, 2015)

Let P_t be the push-forward of vertical polarization by Φ_t^β. For each $\tau > 0$, there is a neighborhood U of zero-section in $T^*(M)$ s.t. the map

$$t \mapsto P_t(z)$$

*analytically continues to $t = i\tau$. The subspaces $P_{i\tau}$ are the $(1,0)$ subspaces of a "**magnetic complex structure**" and this fits with ω^β to define a Kähler structure on $U \subset T^*(M)$.*

Methods of analytic continuation

- Holomorphic functions are still

$$g \circ \pi \circ \Phi_{i\tau}$$

- If M is embedded in X, then $U \subset T^*(M)$ embeds into X as

$$z \mapsto \pi \circ \Phi_{i\tau}(x, p)$$

  via analytic continuation of $t \mapsto \pi \circ \Phi_t(x, p)$
- When $\beta = 0$, we have

$$\pi \circ \Phi_t(x, p) = \exp_x(tp) \quad (\beta = 0 \text{ only})$$

  but in general we must put i in front of τ not p.

Local Kähler potential

- Let A be a local potential for β:

$$dA = \beta$$

- Then define

$$f_\sigma(z) = \sigma E(z) + \int_0^\sigma A\left(\frac{d(\pi \circ \Phi_s)(z)}{ds}\right) ds$$

- Then

$$\kappa(z) = i(f_{-i} - f_i)$$

  is a local Kähler potential for (J^β, ω^β)

- **Note**: The map $(x, v) \mapsto (x, -v)$ is not anti-holomorphic unless $\beta = 0$.

Examples with global magnetic structures

- Plane: $T^*(\mathbb{R}^2)$ bi-holomorphic to $\mathbb{C}^2$
- 2-sphere: $T^*(S^2)$ is then bi-holomorphic to **quadric:**

$$Q^2 := \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \middle| z_1^2 + z_2^2 + z_3^2 = 1 \right\},$$

- Co-adjoint orbits of compact Lie groups

(All with multiple of natural invariant 2-form). But: **identification** depends on normalization of β

Sphere case

- $\beta = B \cdot (\text{area form})$
- Compute **momentum map**

$$\mathbf{J}(\mathbf{x}, \mathbf{p}) = \mathbf{x} \times \mathbf{p} - B\mathbf{x}$$

for action of $SO(3)$ on $(T^*(S^2), \omega^\beta)$

- Isomorphism $f : (T^*(S^2), J_{\text{magnetic}}) \to Q^2$

$$f(\mathbf{x}, \mathbf{p}) = \cosh L \; \mathbf{x} + i \frac{\sinh L}{L} \mathbf{p} - \frac{\cosh L - 1}{L^2} B \; \mathbf{J}(\mathbf{x}, \mathbf{p})$$

where $L = \sqrt{|\mathbf{p}|^2 + B^2}$.

Quantum version of Thiemann's proposal

- **Schrödinger Hilbert space**: using vertical polarization (basically $L^2(M)$)
- **Segal–Bargmann Hilbert space:** using complex polarization (an L^2 space of **holomorphic functions or sections** over $T^*(M)$)
- Seek: unitary map between them using the **quantized Hamiltonian flow**

Compact group case

- $T^*(K)$ with adapted complex structure (non-magnetic) bi-holomorphic to $K_{\mathbb{C}}$
- Quantized geodesic flow (at imaginary time) is **heat operator**
- Heat operator defines **unitary Segal–Bargmann transform** from $L^2(K)$ to $\mathcal{H}L^2(K_{\mathbb{C}}, \nu)$ where ν is a measure coming from geometric quantization [Hall, 1994]
- Can also obtain this map a **unitary BKS pairing map** [Hall, 2002; cf. Florentino, Matias, Mourão, and Nunes, 2005, 2006]

Case of a 2-sphere with magnetic field

- **Schrödinger space** is $\Gamma^2(S^2, L)$, sections of line bundle
- **Segal–Bargmann space** is $\mathcal{H}\Gamma^2(S^2, L^{\mathbb{C}})$, holomorphic sections of line bundle
- Quantized Hamiltonian flow: **bundle heat operator** gives a unitary map [Hall–Mitchell, 2012]
- Also plane case: connects with analysis on Heiseberg group [Krötz, Thangavelu, Xu, 2005]

Other directions

- Other flows on $T^*(K)$ [Kirwin, Mourão, and Nunes, 2014]
- Connections with geodesics in the space of Kähler structures [Mourão, and Nunes, 2015, Burns–Lupercio–Uribe, 2015]

Conclusion

THANK YOU FOR YOUR ATTENTION!