Quantisation of Littlewood-Richardson coefficients beyond type A

AMS-EMS joint meeting

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I. Schur polynomials

$\mathbb{Z}[x_1, \ldots, x_n]$ polynomial ring in $x_1, \ldots, x_n$.
For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}.$$

$\mathbb{Z}^{\mathrm{sym}}[x_1, \ldots, x_n]$ the subring of symmetric polynomials.

Partition of depth d: a non increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d_{>0}$.
For $n \geq d$,

$$s_\lambda = \frac{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(w) x^{\sigma(\lambda+\rho)-\rho}}{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(w) x^{\sigma(\rho)-\rho}} \text{ with } \rho = (n-1, \ldots, 0)$$

is the Schur polynomial associated to λ.

Generalised LR-coefficients

Theorem

1. *s_λ is the character of the f.d. representation $V(\lambda)$ of $\mathfrak{gl}_n(\mathbb{C})$ of highest weight λ.*
2. *$\{s_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a basis of $\mathbb{Z}^{\mathrm{sym}}[x_1, \ldots, x_n]$.*

Let $\mu^{(1)}, \ldots, \mu^{(\ell)}$ be partitions and n s.t.

$$n \geq d(\mu^{(1)}) + \cdots + d(\mu^{(\ell)}).$$

Corollary

The generalised LR-coefficients are the structure constants s.t.

$$s_{\mu^{(1)}} \cdots s_{\mu^{(\ell)}} = \sum_{\lambda \in \mathcal{P}_n} c^{\lambda}_{\mu^{(1)}, \ldots, \mu^{(\ell)}} s_\lambda.$$

Three algebraic interpretations

1. We have

$$V(\mu^{(1)}) \otimes \cdots \otimes V(\mu^{(\ell)}) = \bigoplus_{\lambda \in \mathcal{P}_n} V(\lambda)^{\oplus c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}}.$$

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2. With $n=\sum_k d_k$, we have

$$\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{d_1}\oplus\cdots\oplus\mathfrak{gl}_{d_\ell}}V(\lambda)=\bigoplus_{(\mu^{(1)},\ldots,\mu^{(\ell)})\in\mathcal{P}_{d_1}\times\cdots\times\mathcal{P}_{d_\ell}}V(\mu^{(1)},\ldots,\mu^{(\ell)})^{\oplus c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}}$$

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3. We have

$$\psi_\ell(s_\lambda)=s_\lambda(x_1^\ell,\ldots,x_n^\ell)=\sum_{\mu\in\mathcal{P}_n}\varepsilon(\mu)c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}s_\mu$$

where $\mu^{(1)},\ldots,\mu^{(\ell)}$ is the ℓ-quotient of μ and $\varepsilon(\mu)\in\{-1,0,1\}$.

Quantisation from affine crystals

Let $\lambda = (s^r)$ be a rectangular partition.

Example

A standard tableau of shape $(3,3) = (3^2)$.

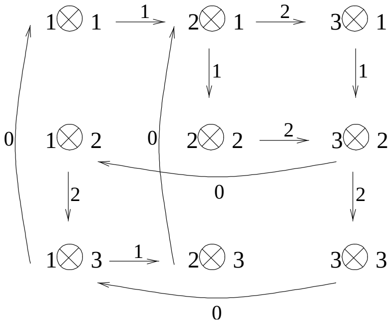
1	2	4
3	3	5

$B_{r,s}$ the set of semistandard tableaux of shape (s^r) has the structure of an affine type A KR crystal.
Any tensor product

$$B := B_{r_1,s_1} \otimes \cdots \otimes B_{r_\ell,s_\ell}$$

has also the structure of an affine connected crystal.
B is graded by the energy D defined from the graph structure.

- D is constant on the classical components (forget the 0 arrows).
- The highest weight vertices are the sources vertices for the c. c.



The affine crystal $B = B_{1,1}^{\otimes 2}$

Definition

$$C^{\lambda}_{s_1^{r_1},\dots,s_\ell^{r_\ell}}(q) = \sum_{v \text{ of h.w. } \lambda} q^{D(v)}$$

where D is the energy.

By construction

$$C^{\lambda}_{s_1^{r_1},\dots,s_\ell^{r_\ell}}(q) = c^{\lambda}_{s_1^{r_1},\dots,s_\ell^{r_\ell}}.$$

Example

For $B_{1,1}^{\otimes 2}$ we get

$$C^{(1,1)}_{(1),(1)} = 1 \text{ and } C^{(2)}_{(1),(1)} = q.$$

Quantisation from the partition function expression

Consider $d = (d_1, \ldots, d_\ell) \in \mathbb{N}^\ell$ with sum n.
Define $I_k = [d_k + 1, d_{k+1}]$ for $k = 1, \ldots, \ell - 1$ and

$$S = \{(i,j) \mid 1 \leq i < j \leq n \text{ and } i,j \text{ belongs to distincts } I_k\}.$$

Set

$$\prod_{(i,j)\in S} \frac{1}{1 - q\frac{x_j}{x_i}} = \sum_{\beta \in \mathbb{Z}^\ell} \mathcal{P}_q^{(d)}(\beta) x^{-\beta}.$$

Definition

For $(\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathcal{P}_{d_1} \times \cdots \times \mathcal{P}_{d_\ell}$ set

$$\mathbf{C}^\lambda_{\mu^{(1)},\ldots,\mu^{(\ell)}}(q) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \mathcal{P}_q^{(d)}(\sigma(\lambda + \rho) - (\mu + \rho))$$

where μ is the concatenation of $\mu^{(1)}, \ldots, \mu^{(\ell)}$.

Theorem

1. *We get*

$$\mathbf{C}^{\lambda}_{\mu^{(1)},\dots,\mu^{(\ell)}}(1) = c^{\lambda}_{\mu^{(1)},\dots,\mu^{(\ell)}}.$$

Conjecture (Broer 1994): When $\mu^{(1)},\dots,\mu^{(\ell)}$ μ is a partition,

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$$\mathbf{C}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}(1) = c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}.$$
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3. *We have moreover $\mathbf{C}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}(q) = C^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}(q)$ (Shimozono 2002).*

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Quantisation from quantum groups at roots of 1

The polynomial s_λ is also the character of $V_q(\lambda)$ the f.d. $U_q(\mathfrak{gl}_n)$-module of h.w. λ.

Let $\xi = \exp(\frac{i\pi}{\ell})$.
Then $V_\xi(\lambda)$ is no longer irreducible but has an irreducible quotient $L(\lambda)$.

Theorem (Kazhdan-Lusztig, Kashiwara-Tanisaki)

We have

$$\psi_\ell(s_\lambda) = \mathrm{char} L(\ell\lambda)$$

and

$$c^\lambda_{\mu^{(1)},\ldots,\mu^{(\ell)}} = P^-_{\mu+\rho,\ell\lambda+\rho}(1)$$

where $P^-_{\mu+\rho,\ell\lambda+\rho}(q)$ is a parabolic K. L. polynomial.

Definition (LLT 1996)

We set

$$\mathfrak{C}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}(q) = P^{-}_{\mu+\rho,\ell\lambda+\rho}(q) \in \mathbb{N}[q].$$

Theorem (Haiman-Grojnoswski 2007 preprint)

When $\mu^{(1)},\ldots,\mu^{(\ell)}$ are rectangular and μ is a partition,

$$\mathfrak{C}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}(q) = \mathbf{C}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(\ell)}}(q).$$

Beyond type A

For type B, C, D, quantisations of branching (or tensor products) coefficients also exist:

- C from the energy function on affine crystals (HKOTZ 1999),

Theorem (L-Okado-Shimozono 2012))

Polynomials C are special cases of polynomials $\mathbf{C}$.

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- $\mathfrak{C}$ from parabolic KL polynomials and plethysms on Weyl characters (L 2007).

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