Triviality Theorems for Yetter-Drinfel'd Hopf Algebras

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First main result

K: Algebraically closed field of characteristic zero (always!)

G: Finite abelian group

A: Yetter-Drinfel'd Hopf algebra over the group ring $K[G]$

Assumption 1: A is commutative and semisimple

Assumption 2: $\dim(A)$ and $|G|$ are relatively prime

Assertion: A is trivial, i.e., an ordinary Hopf algebra

It is therefore the dual group ring $K[H]^*$ of another group H with additional structure making it a Yetter-Drinfel'd module.

Second main result

G: Finite abelian group

A: Yetter-Drinfel'd Hopf algebra over the group ring $K[G]$

Assumption 1: A is cocommutative and semisimple

Assumption 2: $\dim(A) > 1$

Assertion: A contains a trivial Yetter-Drinfel'd

Hopf subalgebra B with $\dim(B) > 1$

Note that $\dim(B)$ divides $\dim(A)$.

The following statement is therefore not contradictory:

If A is nontrivial, then it contains a nontrivial trivial

Yetter-Drinfel'd Hopf subalgebra.

Yetter-Drinfel'd modules

H: Hopf algebra

Yetter-Drinfel'd module: Left module and left comodule over H.

Coaction: $\delta : V \to H \otimes V,\ v \mapsto v^{(1)} \otimes v^{(2)}$

Compatibility condition:

$$\delta(h.v) = h_{(1)} v^{(1)} S(h_{(3)}) \otimes h_{(2)}.v^{(2)}$$

More precisely: Left-left Yetter-Drinfel'd modules

$H = K[G]$: Yetter-Drinfel'd module $=$ G-graded vector space with an additional G-action.

Compatibility condition:

$$\deg(v) = g \Rightarrow \deg(h.v) = hgh^{-1}$$

Quasisymmetry

Tensor product of Yetter-Drinfel'd modules:

Diagonal module and codiagonal comodule structure:

$$h.(v \otimes w) = \Delta(h).(v \otimes w) \qquad \delta(v \otimes w) = v^{(1)}w^{(1)} \otimes v^{(2)} \otimes w^{(2)}$$

$V \otimes W$ and $W \otimes V$ are isomorphic:

$$\sigma_{V,W} : V \otimes W \to W \otimes V, \;\; v \otimes w \mapsto (v^{(1)}.w) \otimes v^{(2)}$$

Note: In contrast to $v \otimes w \mapsto w \otimes v$,

$\sigma_{V,W}$ is H-linear and colinear.

Yetter-Drinfel'd Hopf algebras

Yetter-Drinfel'd Hopf algebra A over H:

Hopf algebra in the category of Yetter-Drinfel'd modules.

This means:

1. A is a (left-left) Yetter-Drinfel'd module over H.
2. A is an ordinary algebra whose product $\mu : A \otimes A \to A$ and unit map $\eta : K \to A,\ \lambda \mapsto \lambda 1$ are H-linear and colinear.
3. A is an ordinary coalgebra whose coproduct $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to K$ are H-linear and colinear.
4. A has an H-linear and colinear antipode S that satisfies the same axioms as for usual Hopf algebras.
5. and ...

The decisive difference

... Δ and ε are algebra homomorphisms.

For the counit, this does not mean anything new.

But when saying that Δ is an algebra homomorphism, we refer to the algebra structure

$$A \otimes A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes \sigma \otimes \mathrm{id}} A \otimes A \otimes A \otimes A \xrightarrow{\mu \otimes \mu} A \otimes A$$

on $A \otimes A$ that uses the quasisymmetry σ, and not the usual flip of the tensor factors.

This algebra structure will be denoted by $A \hat{\otimes} A$.

Trivial Yetter-Drinfel'd Hopf algebras

Consequence: If the quasisymmetry σ coincides with the usual flip of the tensor factors, then a Yetter-Drinfel'd Hopf algebra is an ordinary Hopf algebra.

Converse (P. Schauenburg, New York J. Math. 4 (1998)):
If a Yetter-Drinfel'd Hopf algebra is an ordinary Hopf algebra, then the quasisymmetry σ coincides with the usual flip of the tensor factors.

Such Yetter-Drinfel'd Hopf algebras are called trivial.

However, even trivial Yetter-Drinfel'd Hopf algebras can have interesting Radford biproducts.

Idempotents

A commutative and semisimple $\Rightarrow$

A has a basis of orthogonal primitive idempotents.

Dual basis of A^*: One-dimensional characters.

Every $g \in G$ acts on A via $\phi_g : A \to A$.

This action preserves the homogeneous components.

We turn the coaction into an action of $K[G]^* \cong K[\hat{G}]$,

where $\hat{G} = \mathrm{Hom}(G, K^\times)$ is the character group $\Rightarrow$

Every $\gamma \in \hat{G}$ acts on A via $\psi_\gamma : A \to A$.

Action preserves the homogeneous components $\Rightarrow$

$$\phi_g \circ \psi_\gamma = \psi_\gamma \circ \phi_g$$

Ideals in $A\hat{\otimes}A$

Suppose that e and e' are primitive idempotents
with corresponding characters η and η': $\eta(e) = 1$, $\eta'(e') = 1$.
Define

$$T := \{g \in G \mid \phi_g(e) = e\} \qquad Q' := \{\gamma \in \hat{G} \mid \psi_\gamma(e') = e'\}$$

$$I := \mathrm{Span}(\{\phi_g(e) \mid g \in Q'^{\perp}\}) \qquad I' := \mathrm{Span}(\{\psi_\gamma(e') \mid \gamma \in T^{\perp}\})$$

Proposition: $m := \dim(I) = \dim(I')$ and

1. $I\hat{\otimes}Ke'$ is a minimal left ideal in $A\hat{\otimes}A$.
2. $Ke\hat{\otimes}I'$ is a minimal right ideal in $A\hat{\otimes}A$.
3. $I\hat{\otimes}I'$ is a minimal two-sided ideal in $A\hat{\otimes}A$.

Products of characters

Usually: $\eta\eta' \in A^*$ is not again a character. Instead, we have:

Theorem: There are distinct characters $\omega_1, \ldots, \omega_m$ such that

$$\eta\eta' \in \mathrm{Span}(\omega_1, \ldots, \omega_m)$$

m is the smallest number with this property. In addition, we have

$$\phi_g^*(\eta)\psi_\gamma^*(\eta') \in \mathrm{Span}(\omega_1, \ldots, \omega_m)$$

for all $g \in Q'^\perp$ and all $\gamma \in T^\perp$. (These are m^2 characters.)

Sketch of proof

$I\hat{\otimes}Ke'$ is a left ideal, so an $A\hat{\otimes}A$-module.

By pullback along Δ, it becomes an A-module.

Because A is commutative, it must be the sum of one-dimensional modules $\Rightarrow$

There are characters $\omega_1, \ldots, \omega_m$ and vectors $v_1, \ldots, v_m \in I$ such that

$$\Delta(a)(v_k \otimes e') = \omega_k(a) v_k \otimes e'$$

First special case: $e = e'$

Usually: $S(e)$ is not an idempotent

$\Rightarrow$ The antipode does not map ideals to ideals.

Suppose now that $e = e'$.

Then $T = T'$, $Q = Q'$, but also $I = I'$.

Proposition: $S(I)$ is an ideal.

I the smallest ideal with this property.

Define $G_e := Q^{\perp}/(T \cap Q^{\perp})$.

Then we have $|G_e| = \dim(I) = m$.

We call $|G_e|$ the index of e (or of η).

Corollary: $S(e)$ is an idempotent $\Leftrightarrow$ The index of e is 1

Second special case

Let e be a primitive idempotent.

Let I be the associated ideal just described.

Then we just saw that $S(I)$ is an ideal.

Choose a primitive idempotent $e' \in S(I)$.

The ideal I' associated with e' is $S(I)$.

Then again $T = T'$, $Q = Q'$,

but clearly $I' = S(I) \neq I$ in general.

Theorem

In this situation,

1. one character, say ω_1, is the counit.
2. Unless $m = 1$,

   the index of ω_j is strictly less than m for all $j \leq m$.
3. $\mathrm{Span}(\omega_1, \ldots, \omega_m)$ is a subalgebra of A^*.
4. It is clearly a subcoalgebra of A^*,

   because every ω_i is group-like.
5. $\mathrm{Span}(\omega_1, \ldots, \omega_m)$ is stable under ϕ_g and ψ_γ

   for $g \in Q^\perp$ and $\gamma \in T^\perp$.
6. It is also stable under the antipode.

The core

Span($\omega_1, \ldots, \omega_m$) is called the core of e (or η).

It does not depend on e' as long as $e' \in S(I)$.

So is Span($\omega_1, \ldots, \omega_m$) a Yetter-Drinfel'd Hopf subalgebra of A^*?

No, because it is only stable under ϕ_g and ψ_γ

for $g \in Q^\perp$ and $\gamma \in T^\perp$, and not for all $g \in G$ and $\gamma \in \hat{G}$.

Theorem:

Span($\omega_1, \ldots, \omega_m$) is a Yetter-Drinfel'd Hopf algebra over $K[G_e]$.

Proposition

If $\dim(A) > 1$, then A contains a primitive idempotent of index 1 that is different from the integral.

Proof: This is obvious if every primitive idempotent has index 1.

Otherwise, choose a primitive idempotent e'

of minimal index $m := |G_{e'}| > 1$.

If e is a primitive idempotent that corresponds to a character in the core of e', we have $|G_e| < m$ by the preceding theorem.

Therefore $|G_e| = 1$.

Since there are m such idempotents, one of them is different from the integral.

Proof of the second main result

If e has index 1, define

$$U := \{\eta' \mid T \subset T' \text{ and } Q \subset Q'\}$$

U is stable under G and $\hat{G}$.

$|G_e| = 1$ means $Q^\perp = T \cap Q^\perp$ and therefore $Q^\perp \subset T$.

For two primitive idempotents e' and e'' satisfying $T \subset T'$ and $Q \subset Q'$ as well as $T \subset T''$ and $Q \subset Q''$,

we then have

$$Q''^\perp \subset Q^\perp \subset T \subset T'$$

So $\eta'\eta''$ is again a character.

So U is a group and by Schauenburg's theorem

$B := K[U] \subset A^*$ is a trivial Yetter-Drinfel'd Hopf subalgebra.